On the Birational Section Conjecture over Function Fields

Submitted by Michael Peter Tyler to the University of Exeter
as a thesis for the degree of
Doctor of Philosophy in Mathematics,
August 2017.

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Abstract

The birational variant of Grothendieck's section conjecture proposes a characterisation of the rational points of a curve over a finitely generated field over $\mathbb{Q}$ in terms of the sections of the absolute Galois group of its function field. While the p-adic version of the birational section conjecture has been proven by Jochen Koenigsmann [Koe05], and improved upon by Florian Pop [Pop10], the conjecture in its original form remains very much open. One hopes to deduce the birational section conjecture over number fields from the p-adic version by invoking a local-global principle, but if this is achieved the problem remains to deduce from this that the conjecture holds over all finitely generated fields over $\mathbb{Q}$. This is the problem that we address in this thesis, using an approach which is inspired by a similar result by Mohamed Saïdi [Saï16] concerning the section conjecture for étale fundamental groups. We prove a conditional result which says that, under the condition of finiteness of certain Shafarevich-Tate groups, the birational section conjecture holds over finitely generated fields over $\mathbb{Q}$ if it holds over number fields.

Contents

Introduction

Let X be a smooth, projective, geometrically connected curve over a field k of characteristic zero. Let $\bar{z} : \operatorname{Spec}\Omega \to X$ be a geometric point of X, which determines an algebraic closure $\bar{k}$ of k. Let $G_k := \operatorname{Gal}(k^{\mathrm{sep}}|k)$ denote the absolute Galois group of k, where k^{sep} is the separable closure of k in $\bar{k}$. The *étale fundamental group* $\pi_1(X, \bar{z})$ of X sits in an exact sequence

$$1 \longrightarrow \pi_1(X_{\bar{k}}, \bar{z}) \longrightarrow \pi_1(X, \bar{z}) \longrightarrow G_k \longrightarrow 1$$

where $X_{\bar{k}} := X \times_{\operatorname{Spec} k} \operatorname{Spec}\bar{k}$. By functoriality of the étale fundamental group, a k-rational point $x : \operatorname{Spec} k \to X$ gives rise to a section $s_x : G_k \to \pi_1(X, \bar{z})$ of the above exact sequence, defined up to conjugation by $\pi_1(X_{\bar{k}}, \bar{z})$, with image the decomposition group $D_{\tilde{x}} \subset \pi_1(X, \bar{z})$ of a point above x in some universal pro-étale cover of X.

The section conjecture was proposed by Grothendieck in his 1983 letter to Faltings [Gro97], and asserts the following:

When k is finitely generated over $\mathbb{Q}$ and X is hyperbolic, every section of $\pi_1(X, \bar{z})$ arises as above from a unique k-rational point $x \in X(k)$.

Whether this is true remains very much an open question. Only the uniqueness part of the conjecture is known - that is, no two distinct k-rational points x_1, x_2 give rise to the same conjugacy class of sections of $\pi_1(X, \bar{z})$. This is also known in the more general case when k is a sub-p-adic field [Moc99, Theorem C].

One may also ask whether the assertion holds for other fields. For a given field k, we say the *section conjecture holds over k* if for any smooth, geometrically connected, projective, hyperbolic curve X over k, every section of $\pi_1(X, \bar{z})$ arises from a unique k-rational point $x \in X(k)$.

In his recent paper [Saï16], Mohamed Saïdi considers the following question: suppose we know that the section conjecture holds over a particular field k. Then does it also hold over function fields of transcendence degree 1 over k? Saïdi exhibits conditions on the field k which, under the condition of finiteness of certain Shafarevich-Tate groups (which we will introduce in §2.3), result in an affirmative answer to this question. The class of fields which satisfy these conditions includes, in particular, the finitely generated fields over $\mathbb{Q}$. By induction, therefore, the main result of his paper implies the following:

Assume the section conjecture holds over all number fields. Then, under the condition of finiteness of the above-mentioned Shafarevich-Tate groups, it also holds over finitely generated fields over $\mathbb{Q}$.

Thus, Grothendieck's section conjecture is reduced to the case of number fields, though it is important to stress that this result depends on the finiteness of the Shafarevich-Tate groups. In this thesis, we prove a similar conditional result for the birational variant of the section conjecture.

To a smooth, geometrically connected, projective curve X over a field k, we can associate another group G_X, the absolute Galois group of the function field $k(X)$ of X. This group also sits in an exact sequence

$$1 \longrightarrow G_{X_{\bar{k}}} \longrightarrow G_X \longrightarrow G_k \longrightarrow 1$$

A k-rational point $x \in X(k)$ also defines sections of this sequence, and the image of such a section is contained in the decomposition subgroup $D_{\tilde{x}} \subset G_X$ of some extension of x to $k(X)^{\mathrm{sep}}$. One is naturally led to posit the following, which may be considered a birational analogue of the section conjecture:

When k is finitely generated over $\mathbb{Q}$, every section of G_X arises as above from a unique k-rational point $x \in X(k)$.

Note that this does not require X to be hyperbolic. One may ask whether this assertion holds for other fields. For a given field k, we say the *birational section conjecture holds over k* if for any smooth, geometrically connected, projective curve X over k, every section of G_X arises from a unique k-rational point $x \in X(k)$.

The uniqueness of such a k-rational point x follows from a classical result of F. K. Schmidt - see [NSW08, Corollary 12.1.3]. But the question of existence is still an open problem, though more gains have been made than with Grothendieck's section conjecture. The first notable result was that by Jochen Koenigsmann in [Koe05], where he proves that the birational section conjecture holds over finite extensions of $\mathbb{Q}_p$ and over $\mathbb{R}$. This was more recently refined by Florian Pop in [Pop10], where he proves a $\mathbb{Z}/p$ meta-abelian variant which subsumes the birational p-adic section conjecture.

This leads one to suspect that we may be closer to proving the birational analogue of the section conjecture than we are to proving Grothendieck's original assertion. This motivates the theory of "cuspidalisation", which aims to show that the section conjecture follows from its birational variant (see, for example, [Saï12]). Successful development of such a theory would reinforce the importance of the birational section conjecture.

In this thesis, we consider curves over the same class of fields as introduced in [Saï16], and we prove a similar result reducing the birational section conjecture to the case of number fields. This is conditional on the finiteness of certain Shafarevich-Tate groups, which we define in Definition 2.3.2. Our main results are Theorems A and B in §2.3, which imply the following:

Assume the birational section conjecture holds over all number fields. Then, under the condition of finiteness of the above-mentioned Shafarevich-Tate groups, it also holds over finitely generated fields over $\mathbb{Q}$.

The proof of these Theorems is inspired by the proofs in [Saï16], and proceeds as follows. After specifying the necessary conditions for our base field k, including that the birational section conjecture holds over any finite extension of k, we take a smooth, separated, connected curve C over k and a smooth relative curve $\mathcal{X} \to C$ which satisfies the following properties:

(i) Denoting by K the function field $k(C)$ of C, the generic fibre $X := \mathcal{X} \times_C \operatorname{Spec} K$ is a geometrically connected, hyperbolic curve over K such that $X(K) \neq \emptyset$.

(ii) Denoting by $\mathcal{J} := \operatorname{Pic}^0_{\mathcal{X}/C}$ the relative Jacobian of $\mathcal{X}$, the Tate module $T\text{Ш}(\mathcal{J})$ of the Shafarevich-Tate group of $\mathcal{J}$ is trivial.

The bulk of the exposition is devoted to proving that, in this setting, every section $s : G_K \to G_X$ arises from a K-rational point of X. Using this we easily prove that, under the triviality condition on appropriate Shafarevich-Tate groups, the birational section conjecture holds over any finite extension of K. By induction, therefore, this implies that the birational section conjecture holds over any function field of arbitrary transcendence degree over k.

The proof uses a reduction argument. For each closed point c in C, we investigate whether a given section $s : G_K \to G_X$ specialises to a section of the absolute Galois group $G_{\mathcal{X}_c}$ of the closed fibre $\mathcal{X}_c := \mathcal{X} \times_C \operatorname{Spec} k(c)$. This requires passing to the local setting - taking the completion K_c of K with respect to c and the base change $X_c := X \times_{\operatorname{Spec} K} \operatorname{Spec} K_c$, we define a group $\pi_1(X_c - \tilde{S})$ satisfying the following properties:

(i) There is a natural projection $\pi_1(X_c - \tilde{S}) \twoheadrightarrow G_{K_c}$.

(ii) The section s induces a section $s_c : G_{K_c} \to \pi_1(X_c - \tilde{S})$ of this projection.

(iii) There is a surjective *specialisation homomorphism* $\mathrm{Sp} : \pi_1(X_c - \tilde{S}) \twoheadrightarrow G_{\mathcal{X}_c}$ which makes the following diagram commutative.

$$\begin{tikzcd} \pi_1(X_c - \tilde{S}) \arrow[r] \arrow[d, two heads, "\mathrm{Sp}"'] & G_{K_c} \arrow[l, bend right, "s_c"'] \arrow[d, two heads, "\rho"] \\ G_{\mathcal{X}_c} \arrow[r] & G_{k(c)} \end{tikzcd}$$

(See Theorem 3.3.8.) By investigating the image of the kernel $I_{K_c} = \ker \rho$ under the composite homomorphism $\mathrm{Sp} \circ s_c$, we show that, under our conditions on the base field k, the section s_c is associated to a unique $k(c)$-rational point $\bar{x}_c$ of the closed fibre $\mathcal{X}_c$.

We then investigate the étale abelian sections s^{ab}, s_c^{ab} associated to s and s_c. These correspond to elements of the Galois cohomology groups $H^1(G_K, TJ)$ and $H^1(G_{K_c}, TJ_c)$ respectively, where $J := \mathcal{J} \times_C \operatorname{Spec} K$ denotes the Jacobian of X and $J_c := J \times_{\operatorname{Spec} K} \operatorname{Spec} K_c$ that of X_c. Moreover, there is a natural restriction map $\mathrm{res}_c : H^1(G_K, TJ) \to H^1(G_{K_c}, TJ_c)$, and the image of s^{ab} under the diagonal map

$$H^1(G_K, TJ) \longrightarrow \prod_{c \in C^{\mathrm{cl}}} H^1(G_{K_c}, TJ_c)$$

is precisely the family $(s_c^{\mathrm{ab}})_{c\in C^{\mathrm{cl}}}$. This diagonal map fits into a commutative diagram of *Kummer exact sequences*

$$\begin{tikzcd}
0 \arrow[r] & \widehat{J(K)} \arrow[r] \arrow[d] & H^1(G_K, TJ) \arrow[r] \arrow[d] & TH^1(G_K, J(\overline{K})) \arrow[r] \arrow[d, "\Phi"'] & 0 \\
0 \arrow[r] & \prod_{c\in C^{\mathrm{cl}}} \widehat{J_c(K_c)} \arrow[r] & \prod_{c\in C^{\mathrm{cl}}} H^1(G_{K_c}, TJ_c) \arrow[r] & \prod_{c\in C^{\mathrm{cl}}} TH^1(G_{K_c}, J_c(\overline{K_c})) \arrow[r] & 0
\end{tikzcd}$$

where $\widehat{J(K)} := \varprojlim_N J(K)/NJ(K)$, and similarly for $\widehat{J_c(K_c)}$. (See diagram 4.5.) The kernel of the homomorphism Φ is precisely the Shafarevich-Tate group $T\text{Ш}(\mathcal{J})$, and we show that the assumption $T\text{Ш}(\mathcal{J}) = 0$, together with the existence of the points $\bar{x}_c \in \mathcal{X}_c(k(c))$ associated to the sections s_c, imply that s^{ab} is contained in the image of the inclusion $\widehat{J(K)} \hookrightarrow H^1(G_K, TJ)$. This inclusion may be extended to a sequence of maps

$$X(K) \hookrightarrow J(K) \to \widehat{J(K)} \hookrightarrow H^1(G_K, TJ)$$

Using a result concerning the specialisation of points on abelian varieties [PV10, Proposition 2.4], we show that the map $J(K) \to \widehat{J(K)}$ is injective, and we then use the existence of the points $\bar{x}_c$ to prove that s^{ab} is contained in the image of $X(K)$ under the inclusion $X(K) \hookrightarrow H^1(G_K, TJ)$. That is, s^{ab} corresponds to a K-rational point z of X.

Finally, we use this point z, together with the "limit argument" of Akio Tamagawa [Tam97, Proposition 2.8 (iv)], to prove that our original section $s : G_K \to G_X$ arises from a K-rational point x of X, which necessarily coincides with z.

The layout of this thesis is as follows. In Chapter 1 we will introduce the background theory required for the proofs of our main Theorems. This includes the correspondence between étale covers of smooth, connected, projective curves and finite extensions of their function fields; properties of étale fundamental groups and absolute Galois groups of curves; and sections of geometrically abelian fundamental groups.

In Chapter 2 we introduce the section conjecture and its birational analogue in greater detail. In §2.3 we set up the conditions for our main Theorems. In Definition 2.3.1 we exhibit the conditions that our base field must satisfy. We then give the definition of the relevant Shafarevich-Tate groups in Definition 2.3.2, before stating Theorems A and B.

In Chapter 3 we work in the local setting. We consider a smooth relative curve over the spectrum of a complete discrete valuation ring. We recall the construction of the specialisation homomorphism of fundamental groups as described in [GR71, Exposé X], then show how to extend this to a specialisation homomorphism of absolute Galois groups (Theorem 3.3.8 and diagram 3.7). Using this we consider, in §3.4, the specialisation of sections, and the phenomenon of ramification.

In Chapter 4 we return to the global setting from the statement of Theorems A and B. In §4.1 we explain how to pass to the local setting, and thereby apply the results from Chapter 3. In §4.2 we consider étale abelian sections and show how to apply a local-global principle, during which we encounter the necessity of finiteness of the Shafarevich-Tate group (Lemma 4.2.4 and Corollary 4.2.6).

In Chapter 5 we apply the results of Chapters 3 and 4, together with Tamagawa's limit argument, to prove Theorems A and B.

Acknowledgements

I would like to thank my supervisor, Professor Mohamed Saïdi, without whose guidance and careful explanations this project would have been impossible. I would also like to thank my examiners, Professor Minhyong Kim and Dr Julio Bueno De Andrade, for their close scrutiny of my thesis and their helpful suggestions for improvements.

I would like to thank my second supervisor, Dr Henri Johnston, whose advice helped me steer my research and my career. I am grateful for the many meaningful mathematical discussions I had with the pure maths PhD students at Exeter University. My thanks also go to the Number Theory group at the Oxford Mathematical Institute, who welcomed me into their community and whose seminars and study groups stimulated and broadened my mathematical mind.

Finally, special thanks go to Emma, who, over the last four years, has supported, helped and taught me more than any other.

Notation

Here we introduce some notation which we will use throughout, but which, to simplify the exposition, we will not define explicitly in the text.

- For a scheme X, we will denote the set of closed points of X by X^{cl}.
- Given a ring R and morphisms of schemes $X \to Y$ and $\operatorname{Spec} R \to Y$, we will denote the fibre product $X \times_Y \operatorname{Spec} R$ by X_R.
- For a field k and a given separable closure k^{sep} of k, we will denote the absolute Galois group $\operatorname{Gal}(k^{\mathrm{sep}}|k)$ by G_k.
- For an abelian group A and a positive integer N, we denote by $A[N]$ the group of N-torsion elements of A, that is, the kernel of the "multiplication by N" map $N : A \to A$, $a \mapsto N \cdot a$.
- Given an abelian group A, for any two positive integers M, N such that $N|M$ there is a natural quotient map $\psi_{MN} : A/MA \to A/NA$. Thus there is an inverse system $(A/NA, \psi_{MN})$, and we denote its inverse limit by $\widehat{A}$:

$$\widehat{A} := \varprojlim_{N} A/NA$$

- Given an *abelian variety* A over a field k and an algebraic closure $\bar{k}$ of k, we will denote by TA the Tate module $TA_{\bar{k}}(\bar{k})$ of the abelian group $A_{\bar{k}}(\bar{k})$.
- For a commutative, complete, Noetherian local ring R with maximal ideal $\mathfrak{m}$, we denote by $\operatorname{Spf} R$ the formal spectrum of R, that is, the formal scheme with underlying space $\operatorname{Spec} R$ and structure sheaf $\varprojlim_n \mathcal{O}_{\operatorname{Spec} R} / \mathfrak{m}^n$. In terms of the definition in [Har77, Ch. II, §9, p. 194], $\operatorname{Spf} R$ is the completion of $\operatorname{Spec} R$ along its unique closed point.

Chapter 1

Background

1.1 Basic definitions

Let X be a scheme. For a point $x \in X$, we will denote by $\mathcal{O}_{X,x}$ the stalk of X at x, $\mathfrak{m}_x$ the maximal ideal of $\mathcal{O}_{X,x}$ and $k(x) := \mathcal{O}_{X,x}/\mathfrak{m}_x$ the residue field of X at x. Associated to x, there is a canonical morphism $\operatorname{Spec} k(x) \to X$, which, set-theoretically, maps the unique point of $\operatorname{Spec} k(x)$ to x.

A *geometric point* $\bar{x}$ of X is a morphism $\bar{x} : \operatorname{Spec} \Omega \to X$ where Ω is an algebraically closed field. Denoting by x the image of the morphism $\bar{x}$, Ω is an algebraically closed extension of the residue field $k(x)$ of x. If X is a scheme over a field k then $k(x)$ is an extension of k, hence Ω determines an algebraic closure $\bar{k}$ of k.

Suppose X is an S-scheme. A *section* of the structural morphism $\pi : X \to S$ is a morphism $\alpha : S \to X$ such that $\pi \circ \alpha = \mathrm{Id}$. The set of sections of $X \to S$ will be denoted $X(S)$. When $S = \operatorname{Spec} A$ for some ring A, we also denote $X(S)$ by $X(A)$. Thus a geometric point of X is an element of $X(\Omega)$ for some algebraically closed field Ω, while a point $x \in X$ is canonically associated with an element of $X(k(x))$.

A point $x \in X$ is a *closed point* of X if $\{x\}$ is a Zariski-closed subset of X. Suppose X is a scheme over a field k. In this case, a point $x \in X$ is a closed point if and only if the residue field $k(x)$ is a finite extension of k. For any finite extension l of k, the elements of $X(l)$ are called the *l-rational points* of X.

If l is a finite Galois extension of k, there is a bijection between the set $X(l)$ of l-rational points of X and the set of pairs (x, σ) consisting of a point x with residue field l and an element $\sigma \in \mathrm{Gal}(l|k)$. Via this bijection, if $x \in X$ is a point with residue field l and α denotes the canonical morphism $\operatorname{Spec} l \to X$ associated to x, the pair $(x, 1)$ corresponds to α. An element $\sigma \in \mathrm{Gal}(l|k)$ defines a morphism of k-schemes $\operatorname{Spec}\sigma : \operatorname{Spec} l \to \operatorname{Spec} l$, and the pair (x, σ) corresponds to the l-rational point $\alpha \circ \operatorname{Spec}\sigma : \operatorname{Spec} l \to X$, which also has image x. Thus, for a point $x \in X$ with residue field l, there are $[l : k]$ elements of $X(l)$ with image x.

In particular, there is a bijection between the set $X(k)$ of k-rational points of X and the set of points $x \in X$ such that $k(x) = k$. Via this bijection, we will identify an element $\alpha \in X(k)$ with its image $x \in X$.

Definition 1.1.1. Let X be a scheme.

(i) Let x, y be two points of X. We say that x *specialises to* y if $y \in \overline{\{x\}}$.

(ii) A point $x \in X$ is a *generic point* if x is the unique point which specialises to x.

Let X be a scheme. There is a bijection between the irreducible components of X and the generic points of X, with each irreducible component being of the form $\overline{\{\xi\}}$ for some generic point ξ of X.

Assume X is an integral scheme. Then it has a unique generic point ξ, and for any open subset $U \subset X$ and any point $x \in U$ the canonical homomorphisms

$$\mathcal{O}_X(U) \to \mathcal{O}_{X,x} \to \mathcal{O}_{X,\xi}$$

are injective. If U is affine, these homomorphisms induce isomorphisms:

$$\mathrm{Frac}(\mathcal{O}_X(U)) \simeq \mathrm{Frac}(\mathcal{O}_{X,x}) \simeq \mathcal{O}_{X,\xi}$$

The field $\mathcal{O}_{X,\xi}$ is called the *function field* of X, and will be denoted $k(X)$.

A geometric point $\bar{\xi} : \operatorname{Spec}\Omega \to X$ whose image is the generic point ξ determines an algebraic closure $\overline{k(X)}$ of $k(X)$. If X is a scheme over a field k then $\bar{\xi}$ also determines an algebraic closure $\bar{k}$ of k.

Definition 1.1.2. An *algebraic variety* is a scheme of finite type over a field k. An *algebraic curve* over k is an algebraic variety over k whose irreducible components are of dimension 1.

An algebraic variety over k is thus a Noetherian scheme having a cover by finitely many affine open subschemes which are the spectra of finitely generated k-algebras.

Let $X \to Y$ be a morphism of finite type. For a point $y \in Y$, denote by $X_y := X \times_Y \operatorname{Spec} k(y)$ the fibre above y. Since morphisms of finite type are stable under base change, the morphisms $X_y \to \operatorname{Spec} k(y)$, for each $y \in Y$, are also of finite type. Thus the fibres X_y are algebraic varieties over their respective fields $k(y)$. In this way, X can be thought of as a family of algebraic varieties, parameterised by the points of Y.

Definition 1.1.3. Let S be a Dedekind scheme. A *fibred surface* over S is an integral, proper, flat S-scheme $\pi : X \to S$ of dimension 2. We will be interested in the case where S has dimension 1, in which case X may be called a *relative curve* over S, or an *S-curve.*

If $\eta \in S$ denotes the generic point of S, the fibre X_η above η is called the *generic fibre* of the fibred surface X. For a closed point $s \in S$, the fibre X_s above s is called a *closed fibre* of X.

Suppose $S = \operatorname{Spec} R$ where R is a complete discrete valuation ring, and let $X \to S$ be a relative curve over S. Then S has a unique closed point s, thus a unique closed fibre X_s. For any closed point $x \in X_\eta^{\mathrm{cl}}$, the Zariski closure $\overline{\{x\}}$ of x in X is a local scheme, whose closed point is the unique point of $\overline{\{x\}} \cap X_s$. The map

$$r : X_\eta^{\mathrm{cl}} \to X_s$$

which takes $x \in X_\eta^{\mathrm{cl}}$ to the unique point of $\overline{\{x\}} \cap X_s$ is called the *reduction map* of X. For $x \in X_\eta^{\mathrm{cl}}$, the image $r(x)$ is called the *specialisation* of x, and we say x *specialises* to $r(x)$. Regarding x and $r(x)$ as points of X, this means x specialises to $r(x)$ in the sense of Definition 1.1.1. The reduction map is surjective onto the set X_s^{cl} of closed points of X_s.

Definition 1.1.4. Let C be a smooth, connected, projective algebraic curve over a field k, and S a Dedekind scheme of dimension 1 having function field k. A *model of C over S* is a normal fibred surface $X \to S$ with an isomorphism $X_\eta \simeq C$.

For a scheme X, a point $x \in X$ is called a *point of codimension 1* if the subset $\overline{\{x\}} \subset X$ has codimension 1 in X.

Let X be an integral, normal, locally Noetherian scheme with function field $k(X)$. For any point $x \in X$ of codimension 1, the local ring $\mathcal{O}_{X,x}$ is normal of dimension 1, and thus a discrete valuation ring, with field of fractions $k(X)$. Denoting by t_x

a uniformiser of $\mathcal{O}_{X,x}$, any element $f \in \mathcal{O}_{X,x}$ can be uniquely written in the form $f = t_x^n u$, where u is a unit in $\mathcal{O}_{X,x}$ and $n \in \mathbb{N}$. We define a discrete valuation ν_x on $\mathcal{O}_{X,x}$ by $\nu_x(f) = n$, and a valuation on $k(X)$ by $\nu_x(f/g) = \nu_x(f) - \nu_x(g) \in \mathbb{Z}$. Thus the codimension 1 points of X give rise to discrete valuations on $k(X)$.

In particular, if X is an integral normal algebraic curve over a field k, the points of codimension 1 are simply the closed points. Thus every closed point $x \in X$ defines a discrete valuation on $k(X)$. If X is also projective, then the set of closed points of X is in bijection with the set of discrete valuations on $k(X)$ which are trivial on k.

Definition 1.1.5. Let X be an integral, normal, Noetherian scheme.

(i) A *divisor* on X is a formal sum

$$\sum_{x \in X,\, \dim \mathcal{O}_{X,x}=1} n_x[\overline{\{x\}}]$$

over the codimension 1 points of X, where $n_x = 0$ for all but finitely many such x. When X is a curve we will write a divisor on X as

$$\sum_{x \in X \text{ closed}} n_x[x]$$

(ii) A *principal divisor* on X is a divisor of the form

$$(f) := \sum_{x \in X,\, \dim \mathcal{O}_{X,x}=1} \nu_x(f)[\overline{\{x\}}]$$

for $f \in k(X)$ a rational function and ν_x the discrete valuation associated to x.

(iii) When X is a curve over a field k, we define the *degree* of a divisor $D = \sum n_x[x]$ on X to be the integer

$$\deg_k D := \sum_{x \in X \text{ closed}} n_x[k(x) : k]$$

Remark 1.1.6. For an integral, normal, Noetherian scheme X and a finite set $\{x_1, \ldots, x_r\}$ of codimension 1 points of X, we will identify the union $\overline{\{x_1\}} \cup \cdots \cup \overline{\{x_r\}}$ with the divisor $\sum_{i=1,\ldots,r} [\overline{\{x_i\}}]$.

We denote the group of divisors on X by $\operatorname{Div}(X)$. Since ν_x is additive for each point x of codimension 1, the principal divisors form a subgroup of $\operatorname{Div}(X)$.

Definition 1.1.7. Let X be an integral, normal algebraic curve. The *Picard group* $\mathrm{Pic}(X)$ is the quotient of $\mathrm{Div}(X)$ by the subgroup of principal divisors on X. We denote by $\mathrm{Pic}^0(X)$ the subgroup of $\mathrm{Pic}(X)$ of classes of divisors of degree zero.

Theorem 1.1.8. *Let X be a smooth, geometrically connected, projective curve over a field k, and let g denote the genus of X. Then there exists an abelian variety J of dimension g over k such that, for every extension $l|k$ with $X(l) \neq \emptyset$, there is an isomorphism $J(l) \simeq \mathrm{Pic}^0(X_l)$, where $X_l := X \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, l$.*

Proof. See [Liu02, Theorem 7.4.39]. □

Definition 1.1.9. The abelian variety J in the above Theorem is called the *Jacobian of X*.

Remark 1.1.10. The notion of the Jacobian can be extended to relative curves via the relative Picard functor [BLR90, Definition 8.1.2]. Given a smooth relative curve $X \to S$ with geometrically connected fibres, the identity component of the relative Picard functor $\mathrm{Pic}_{X/S}$ is representable by an abelian S-scheme [BLR90, Proposition 9.4.4], which we will denote by $\mathrm{Pic}^0_{X/S}$ and which we will refer to as the *relative Jacobian of X*. For such a relative curve, the relative Jacobian $\mathrm{Pic}^0_{X/S}$ is a Néron model of its generic fibre, which is the Jacobian of the generic fibre of X [BLR90, Proposition 1.2.4 and Theorem 9.5.1].

1.2 Projective curves and field extensions

Lemma 1.2.1. *Let $f : Y \to X$ be a morphism of projective curves over a field k. The following conditions are equivalent.*

(i) *f is finite.*

(ii) *For any irreducible component Y_i of Y, $f(Y_i)$ is not reduced to a point.*

(iii) *For ξ any generic point of Y, $f(\xi)$ is a generic point of X.*

Proof. See [Liu02, Lemma 7.3.10]. □

Corollary 1.2.2. *Let X, Y be irreducible projective curves over k with generic points ξ_X, ξ_Y respectively, and let $f : Y \to X$ be a morphism. The following conditions are equivalent.*

(i) *f is finite.*

(ii) *f is surjective.*

(iii) $f(\xi_Y) = \xi_X$.

Let X, Y be integral projective curves over k with generic points ξ_X, ξ_Y respectively, and let $f : Y \to X$ be a finite morphism. By Corollary 1.2.2 we have $f(\xi_Y) = \xi_X$, hence we have a canonical homomorphism $\mathcal{O}_{X,\xi_X} \to \mathcal{O}_{Y,\xi_Y}$, that is, an extension of function fields

$$k(X) \hookrightarrow k(Y)$$

which is a finite extension since f is finite. Thus, to a finite morphism of integral projective curves over k is associated a finite extension of function fields in one variable over k. Conversely, one can associate to such a field extension a finite morphism of integral projective curves, thanks to the following Proposition.

Proposition 1.2.3. *Let k be a field. Then for any function field K in one variable over k, there exists, up to isomorphism, a unique integral, normal, projective curve X over k such that $k(X) = K$.*

Proof. See [Liu02, Proposition 7.3.13 (a)]. □

Let K be a function field in one variable over k, and X^K its associated integral, normal, projective curve. Let $L|K$ be a finite extension. Then the normalisation $X^L \to X^K$ of X^K in L is a finite morphism. This establishes a correspondence between finite morphisms of curves and finite extensions of function fields in one variable over k.

Theorem 1.2.4. *Let k be a field. There exists an equivalence of categories between the category of integral, normal, projective curves over k with finite morphisms and the category of finitely generated field extensions of k of transcendence degree 1 with morphisms being field extensions.*

$$\left\{\begin{array}{l}\text{Integral, normal,}\\ \text{projective curves over } k\\ \text{with finite morphisms}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{Finitely generated extensions}\\ \text{of } k \text{ of transcendence degree}\\ \text{1 with field extensions}\end{array}\right\}$$

$$\begin{array}{ccc} Y \longrightarrow X & \longmapsto & k(Y)|k(X) \\ X^L \longrightarrow X^K & \longleftarrow\!\!\shortmid & L|K \end{array}$$

The following Proposition shows that this equivalence also characterises integral, normal curves which are not projective.

Proposition 1.2.5. (i) *Every integral, normal curve U can be embedded as an open subset in a unique integral, normal, projective curve X.*

(ii) *Every morphism $V \to U$ of integral, normal curves extends uniquely to a morphism $Y \to X$ of integral, normal, projective curves.*

Proof. See [Sza09, Proposition 4.4.6] □

1.3 Étale covers

All schemes in this section are assumed to be locally Noetherian.

Definition 1.3.1. Let $f : Y \to X$ be a morphism of finite type, y a point of Y, and $x := f(y)$. We say f is *unramified* at y if the homomorphism $\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ satisfies $\mathfrak{m}_x \mathcal{O}_{Y,y} = \mathfrak{m}_y$ and the extension of residue fields $k(y)|k(x)$ is separable. Otherwise we say f is *ramified* at y.

Definition 1.3.2. Let $f : Y \to X$ be a morphism of finite type, and let $y \in Y$. We say f is *étale at y* if it is flat and unramified at y. If $x \in X$ is a point such that f is étale at every point $y \in Y$ with $f(y) = x$, we say f is *étale over x*.

We say f is *étale* if it is étale at every point of Y. If f is étale and finite, we call f an *étale cover* of X.

Suppose X and Y are integral, normal, projective curves over a field k, and let $f : Y \to X$ be a finite morphism. Let y be a point of Y and $x := f(y)$ its image in X. Then the canonical homomorphism $\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ induced by f is an extension of discrete valuation rings. Thus f is unramified at y if and only if this extension of rings, or equivalently the extension of function fields $k(Y)|k(X)$, is unramified in the classical sense with respect to the valuation ν_y defined by y.

Moreover, the homomorphism $\mathcal{O}_{X,x} \hookrightarrow \mathcal{O}_{Y,y}$ is flat by [Liu02, Corollary 1.2.5]. Hence, to check that f is étale at y, it suffices to check it is unramified at y.

Lemma 1.3.3. *Let $f : Y \to X$ be a morphism of finite type. The set of points $y \in Y$ at which f is étale is open in Y.*

Proof. See [Liu02, Corollary 4.4.12]. □

Lemma 1.3.4. *Let X be a regular, connected scheme, and let $f : Y \to X$ be a finite, flat morphism with Y irreducible. Then the set of points $x \in X$ over which f is étale is open in X.*

Proof. For any $x \in X$ and any $y \in f^{-1}(x)$, f is étale at y if and only if it is smooth at y [Liu02, Theorem 6.2.7 and Corollary 6.2.3]. The Lemma then follows from [Gro67, Ch. IV, Théorème 12.2.4 (iii)]. $\square$

Let $f : Y \to X$ be a finite, flat morphism with X regular and connected and Y irreducible. By Lemma 1.3.4, the set of points $U \subset X$ over which f is étale is open in X. The complement $S := X - U$ is called the *branch locus* of the cover $f : Y \to X$. The points of the branch locus are called the *branch points* of f.

Theorem 1.3.5 (Zariski's Purity Theorem)**.** *Let $f : Y \to X$ be a finite, surjective morphism of integral schemes with X regular and Y normal, and denote by S the branch locus of f. Then either the irreducible components of S have codimension 1, or $S = \emptyset$.*

Proof. See [Sza09, Theorem 5.2.13] or [GR71, Exposé X, Theorem 3.1]. $\square$

Definition 1.3.6. For a morphism of schemes $f : Y \to X$, we define $\mathrm{Aut}(Y|X)$ to be the group of automorphisms of Y which preserve f, that is, automorphisms $\varphi : Y \to Y$ such that $f \circ \varphi = f$.

For the remainder of this section, $f : Y \to X$ will denote a finite morphism of smooth, geometrically connected, projective curves over a field k. Let y be a closed point of Y, and denote $x := f(y)$. The extension of discrete valuation rings $\mathcal{O}_{X,x} \hookrightarrow \mathcal{O}_{Y,y}$ verifies $\mathfrak{m}_x \mathcal{O}_{Y,y} = \mathfrak{m}_y^{e_y}$. The integer e_y is called the *ramification index* of f at y. Clearly, f is unramified at y if and only if $e_y = 1$ and the extension of residue fields $k(y)|k(x)$ is separable.

An element $\sigma \in \mathrm{Aut}(Y|X)$ defines an automorphism $\sigma : Y \to Y$, hence in particular a homomorphism $\sigma_y : \mathcal{O}_{Y,\sigma(y)} \to \mathcal{O}_{Y,y}$. If $\sigma(y) = y$ then σ_y is an automorphism of $\mathcal{O}_{Y,y}$, which also defines an automorphism of $k(y)$.

Definition 1.3.7. With the above notation, assume further that the extension of function fields $k(Y)|k(X)$ is Galois. The *decomposition group* D_y of y is the stabiliser of y in $\mathrm{Aut}(Y|X)$. By the above paragraph, there is a natural action of D_y on $k(y)$, and thus a homomorphism $D_y \to \mathrm{Aut}(k(y)|k(x))$. The *inertia group* I_y is the kernel of this homomorphism.

Under the conditions of this Definition, Theorem 1.2.4 implies that the automorphism group $\mathrm{Aut}(Y|X)$ is isomorphic to $\mathrm{Aut}(k(Y)|k(X))$. Thus, when $k(Y)|k(X)$ is a Galois extension, the decomposition and inertia groups D_y, I_y are isomorphic to the decomposition and inertia subgroups of the Galois group $\mathrm{Gal}(k(Y)|k(X))$ associated to the extension of valuations $\nu_y|\nu_x$ corresponding to the points y and x.

Lemma 1.3.8. *With the notation and assumptions of Definition 1.3.7, the sequence*

$$1 \longrightarrow I_y \longrightarrow D_y \longrightarrow \mathrm{Aut}(k(y)|k(x)) \longrightarrow 1$$

is exact.

Proof. By the above paragraph, this follows from [Neu99, Ch. II, Proposition 9.9]. □

Proposition 1.3.9. *With the notation and assumptions of Definition 1.3.7, the morphism f is étale at y if and only if the inertia group I_y is trivial. If $k(x)$ has characteristic zero, then f is ramified at y if and only if the ramification index e_y of f at y is greater than 1 and I_y is canonically isomorphic to the group μ_{e_y} of e_y-th roots of unity.*

Proof. As above, the inertia group I_y is isomorphic to the inertia subgroup of $\mathrm{Gal}(k(Y)|k(X))$ corresponding to the extension of valuations $\nu_y|\nu_x$. The cover f is étale at y if and only if it is unramified at y (see the discussion after Definition 1.3.2), and this occurs exactly when the extension $k(Y)|k(X)$ is unramified with respect to ν_y, in which case I_y is trivial. The second statement follows from [Lan86, Ch. 2, §5, Proposition 12]. □

1.4 The fundamental group and absolute Galois group

All schemes in this section are assumed to be locally Noetherian.

For a scheme X, denote by Et_X the category whose objects are étale covers of X, and whose morphisms are morphisms of schemes over X. Fix a geometric point $\bar{z} : \mathrm{Spec}\,\Omega \to X$. For an object $f : Y \to X$ in Et_X, we call the fibre $Y_{\bar{z}} = Y \times_X \mathrm{Spec}\,\Omega$ the *geometric fibre* above $\bar{z}$. Define $\mathrm{Fib}_{\bar{z}}(Y)$ to be the underlying set of the geometric fibre $Y_{\bar{z}}$, which is a finite set since the morphism f is finite. A morphism $Y \to Y'$ in Et_X induces a morphism $Y_{\bar{z}} \to Y'_{\bar{z}}$ of geometric fibres, hence a set-theoretic map

$\mathrm{Fib}_{\bar{z}}(Y) \to \mathrm{Fib}_{\bar{z}}(Y')$. Thus $\mathrm{Fib}_{\bar{z}}$ is a functor from Et_X to the category of sets, which we call the *fibre functor* at $\bar{z}$.

By base change, there is a natural action of $\mathrm{Aut}(Y|X)$ on $Y_{\bar{z}}$. If Y is connected then, by [Sza09, Corollary 5.3.4], $\mathrm{Aut}(Y|X)$ acts on $Y_{\bar{z}}$ without fixed points, which in particular implies that $\mathrm{Aut}(Y|X)$ is finite.

Definition 1.4.1. Let X be a connected scheme and $\bar{z} : \operatorname{Spec} \Omega \to X$ a geometric point. An étale cover $f : Y \to X$ is called an étale *Galois* cover if Y is connected and $\mathrm{Aut}(Y|X)$ acts transitively on $\mathrm{Fib}_{\bar{z}}(Y)$.

Lemma 1.4.2. *Let X be a connected scheme and $\bar{z} : \operatorname{Spec} \Omega \to X$ a geometric point. The fibre functor $\mathrm{Fib}_{\bar{z}}$ is pro-representable, that is, there exists an inverse system (Y_i, φ_{ij}) of objects of Et_X, indexed by a partially ordered set I, such that for any object Y in Et_X there is a functorial isomorphism*

$$\mathrm{Fib}_{\bar{z}}(Y) \simeq \varinjlim_{i} \mathrm{Hom}(Y_i, Y)$$

One can take the inverse system (Y_i, φ_{ij}) which pro-represents $\mathrm{Fib}_{\bar{z}}$ to be system of all étale Galois covers $Y_i \to X$. We refer to [Sza09, Proposition 5.4.6] for a proof.

An automorphism of a functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ between two categories is a morphism of functors $F \to F$ that has a two-sided inverse. Giving an automorphism of F amounts to giving, for each object X in $\mathcal{C}_1$, an automorphism $\gamma_X \in \mathrm{Aut}(F(X))$ such that for any morphism $f : X \to Y$ in $\mathcal{C}_1$ the following diagram commutes.

$$\begin{array}{ccc} F(X) & \xrightarrow{\gamma_X} & F(X) \\ \big\downarrow{\scriptstyle F(f)} & & \big\downarrow{\scriptstyle F(f)} \\ F(Y) & \xrightarrow{\gamma_Y} & F(Y) \end{array}$$

The automorphism group $\mathrm{Aut}(F)$ is the set of automorphisms of F with group operation being composition of morphisms.

Definition 1.4.3. Let X be a connected scheme and $\bar{z} : \operatorname{Spec} \Omega \to X$ a geometric point. The *étale fundamental group* $\pi_1(X, \bar{z})$ of X is the automorphism group $\mathrm{Aut}(\mathrm{Fib}_{\bar{z}})$ of the fibre functor on Et_X at $\bar{z}$.

Theorem 1.4.4. *Let X be a connected scheme and $\bar{z} : \operatorname{Spec} \Omega \to X$ a geometric point. The functor $\mathrm{Fib}_{\bar{z}}$ induces an equivalence of Et_X with the category of finite sets with a continuous (with respect to the discrete topology) left action of $\pi_1(X, \bar{z})$.*

The objects $Y \to X$ of Et_X with Y connected correspond to sets with transitive $\pi_1(X, \bar{z})$-action. The Galois covers $Y \to X$ correspond to finite quotients of $\pi_1(X, \bar{z})$.

Proof. See [Sza09, Theorem 5.4.2]. □

Under this equivalence, an étale cover $Y \to X$ with Y connected corresponds to an open subgroup of $\pi_1(X, \bar{z})$, while an étale Galois cover $Y \to X$ corresponds to an open normal subgroup $N \subset \pi_1(X, \bar{z})$, with $\pi_1(X, \bar{z})/N \simeq \mathrm{Aut}(Y|X)$.

Lemma 1.4.5. *With the notation of Theorem 1.4.4, let (Y_i, φ_{ij}) be the inverse system which pro-represents the fibre functor $\mathrm{Fib}_{\bar{z}}$, consisting of all the étale Galois covers of X. Then the automorphism groups $\mathrm{Aut}(Y_i|X)$ form an inverse system, and we have an isomorphism*

$$\pi_1(X, \bar{z}) \simeq \varprojlim_i \mathrm{Aut}(Y_i|X)$$

In particular, the fundamental group is a profinite group.

Proof. See [Sza09, Corollary 5.4.8]. □

Remark 1.4.6. Et_X is an example of a *Galois category*, which is a category with certain conditions which permit the notions of Galois objects and fundamental group. See [GR71, Exposé V] for further information on Galois categories.

Example 1.4.7. Let X be a connected scheme of finite type over $\mathbb{C}$. One can associate to X a complex analytic space X^{an}, which is the set $X(\mathbb{C})$ endowed with the analytic topology. For some choice of base point $z \in X^{\mathrm{an}}$, this space has an associated topological fundamental group $\pi_1^{\mathrm{top}}(X^{\mathrm{an}}, z)$. There is a canonical morphism $\alpha : X^{\mathrm{an}} \to X$ and an isomorphism

$$\widehat{\pi_1^{\mathrm{top}}(X^{\mathrm{an}}}, z) \simeq \pi_1(X, \bar{z}) \tag{1.1}$$

where $\widehat{\pi_1^{\mathrm{top}}(X^{\mathrm{an}}}, \bar{z})$ denotes the profinite completion of the topological fundamental group of X^{an} and $\bar{z} : \mathrm{Spec}\,\Omega \to X$ is a geometric point with image $\alpha(z)$ [GR71, Exposé XII, Corollaire 5.2 and Théorème et définition 1.1]. This follows from Riemann's Existence Theorem [GR71, Exposé XII, Théorème 5.1], which says there is an equivalence of categories between the étale covers of X and those of the analytic space X^{an}.

Suppose X is a smooth, connected, projective curve of genus g over $\mathbb{C}$. Let $U \subset X$ be an open subset with complement $S := X - U$, and denote $r := \deg_{\mathbb{C}} S$, where S is regarded as a divisor on X (see Remark 1.1.6). Define a group $\Pi_{g,r}$ by the presentation

$$\Pi_{g,r} := \frac{\langle a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_r \rangle}{\langle a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} c_1 \cdots c_r = 1 \rangle}$$

There is an isomorphism $\pi_1^{\mathrm{top}}(U^{\mathrm{an}}, \bar{z}) \simeq \Pi_{g,r}$ between the topological fundamental group of the analytic space U^{an} and the finitely generated group $\Pi_{g,r}$ (see [Sza09, Remark 3.6.4]). In light of isomorphism 1.1, we thus have an isomorphism

$$\pi_1(U, \bar{z}) \simeq \widehat{\Pi}_{g,r} \tag{1.2}$$

where $\widehat{\Pi}_{g,r}$ denotes the profinite completion of $\Pi_{g,r}$. This implies that $\pi_1(U, \bar{z})$ is abelian if and only if $(g, r) = (0, 0), (0, 1), (0, 2)$ or $(1, 0)$, which is to say the quantity $2 - 2g - r$ is non-negative.

More generally, isomorphism 1.2 holds when X is a smooth, connected, projective curve over any algebraically closed field k of characteristic 0. Indeed, there exists a finitely generated field K over $\mathbb{Q}$ and a smooth, projective curve V over K such that $U = V_k$. The field K is isomorphic to a subfield of $\mathbb{C}$, hence, denoting by $\overline{K}$ the algebraic closure of K in k, there exist embeddings $\overline{K} \hookrightarrow \mathbb{C}$ and $\overline{K} \hookrightarrow k$. We therefore have morphisms $V_{\mathbb{C}} \to V_{\overline{K}}$ and $U \to V_{\overline{K}}$, which give rise by functoriality to homomorphisms $\pi_1(V_{\mathbb{C}}, \bar{z}') \to \pi_1(V_{\overline{K}}, \bar{z}'')$ and $\pi_1(U, \bar{z}) \to \pi_1(V_{\overline{K}}, \bar{z}'')$, for appropriate choices of $\bar{z}'$ and $\bar{z}''$. These homomorphisms are isomorphisms by [GR71, Corollaire 1.8], hence we have isomorphisms

$$\pi_1(U, \bar{z}) \simeq \widehat{\pi_1^{\mathrm{top}}(V_{\mathbb{C}}^{\mathrm{an}}, z')} \simeq \widehat{\Pi}_{g,r}$$

Definition 1.4.8. Let X be a smooth, geometrically connected, projective curve of genus g over a field k of characteristic zero, and let $U \subset X$ be an open subset with complement $S := X - U$. Let $\bar{z} : \operatorname{Spec} \Omega \to U$ be a geometric point with image in U, and let $\bar{k}$ denote the algebraic closure of k in Ω. We say the curve U is *hyperbolic* if the fundamental group $\pi_1(U_{\bar{k}}, \bar{z})$ is non-abelian. By the above discussion, this occurs exactly when $2 - 2g - \deg_k S < 0$, where S is regarded as a divisor on X.

Example 1.4.9. Let us consider the spectrum of a field, $\operatorname{Spec} k$, with geometric point $\bar{z} : \operatorname{Spec} \Omega \to \operatorname{Spec} k$. Let $\bar{k}$, respectively k^{sep} denote the algebraic, resp. separable closure of k in Ω. A connected étale cover takes the form $\operatorname{Spec} l \to \operatorname{Spec} k$, where l is a finite separable extension of k. The fibre $\mathrm{Fib}_{\bar{z}}(\operatorname{Spec} l)$ is the underlying set of $\operatorname{Spec}(l \otimes_k \bar{k})$, or in other words, the set of geometric points $\operatorname{Spec} \Omega \to \operatorname{Spec} l$, which is in bijection with the set of k-algebra homomorphisms $l \to \Omega$. Since $l|k$ is a separable extension, such homomorphisms have image in the separable closure k^{sep} of k in Ω, so we have $\mathrm{Fib}_{\bar{z}}(\operatorname{Spec} l) \simeq \mathrm{Hom}_k(l, k^{\mathrm{sep}})$. Thus $\mathrm{Aut}(\mathrm{Fib}_{\bar{z}}) = \mathrm{Aut}(\mathrm{Hom}_k(-, k^{\mathrm{sep}})) = \mathrm{Gal}(k^{\mathrm{sep}}|k)$, where, for any finite separable extension $l|k$, an element $\sigma \in \mathrm{Gal}(k^{\mathrm{sep}}|k)$ defines an automorphism of $\mathrm{Hom}_k(l, k^{\mathrm{sep}})$ by $\phi \mapsto \sigma \circ \phi$. Thus we have an isomorphism:

$$\pi_1(\operatorname{Spec} k, \bar{z}) \simeq G_k := \mathrm{Gal}(k^{\mathrm{sep}}|k)$$

The geometric point $\bar{z}$ in the definition of the étale fundamental group is called the *base point*. The fundamental group depends on the choice of base point, since this is instrumental in the definition of the fibre functor. However, different choices of base point give rise to isomorphic groups, as explained in the following theorem (which may be found in [Sza09, Corollary 5.5.2]).

Theorem 1.4.10. *Let X be a connected scheme. For any two geometric points $\bar{z} : \operatorname{Spec} \Omega \to X$ and $\bar{z}' : \operatorname{Spec} \Omega' \to X$, there exists a continuous isomorphism of étale fundamental groups $\pi_1(X, \bar{z}) \simeq \pi_1(X, \bar{z}')$ which is unique up to inner automorphism of $\pi_1(X, \bar{z}')$.*

This follows from the fact that there exists an isomorphism of fibre functors $\lambda : \mathrm{Fib}_{\bar{z}} \xrightarrow{\sim} \mathrm{Fib}_{\bar{z}'}$ ([Sza09, Proposition 5.5.1]), which defines an isomorphism of their automorphism groups via $\sigma \mapsto \lambda^{-1} \circ \sigma \circ \lambda$. Such an isomorphism of fibre functors is called a *path* from $\bar{z}$ to $\bar{z}'$, and composing it with an inner automorphism of $\pi_1(X, \bar{z}')$ defines another such path. Thus, in analogy with the topological situation, the isomorphisms in the above theorem may be regarded as coming from a choice of path from $\bar{z}$ to $\bar{z}'$.

Let $\varphi : X \to X'$ be a morphism of connected schemes, and let $\bar{z} : \operatorname{Spec} \Omega \to X$ and $\bar{z}' : \operatorname{Spec} \Omega \to X'$ be geometric points such that $\varphi \circ \bar{z} = \bar{z}'$. The morphism φ induces a base change functor $\mathrm{BC}_{X',X} : \mathrm{Et}_{X'} \to \mathrm{Et}_X$, taking an étale cover

$Y' \to X$ to the base change $Y' \times_{X'} X \to X$. Since $\varphi \circ \bar{z} = \bar{z}'$ we have an equality $\mathrm{Fib}_{\bar{z}} \circ \mathrm{BC}_{X',X} = \mathrm{Fib}_{\bar{z}'}$, hence an automorphism of $\mathrm{Fib}_{\bar{z}}$ induces one of $\mathrm{Fib}_{\bar{z}'}$ by composition with $\mathrm{BC}_{X',X}$. The morphism φ therefore induces a canonical continuous homomorphism of profinite groups

$$\varphi_* : \pi_1(X, \bar{z}) \to \pi_1(X', \bar{z}')$$

Thus the fundamental group is functorial with respect to morphisms which preserve base points. If we are given a morphism $\varphi : X \to X'$ which does not satisfy $\varphi \circ \bar{z} = \bar{z}'$, then we may compose φ_* with an appropriate isomorphism, as in Theorem 1.4.10, to give a homomorphism $\varphi_* : \pi_1(X, \bar{z}) \to \pi_1(X', \bar{z}')$ defined up to inner automorphism of $\pi_1(X', \bar{z}')$.

Proposition 1.4.11. (i) *The homomorphism φ_* is surjective if and only if for every étale cover $Y' \to X'$ with Y' connected, the base change $Y' \times_{X'} X$ is also connected.*

(ii) *If every connected étale cover of X is of the form $Y' \times_{X'} X \to X$ for some étale cover $Y' \to X'$, then φ_* is injective.*

Proof. See [Sza09, Proposition 5.5.4 (2) and Corollary 5.5.8]. □

Definition 1.4.12. A *universal pro-étale cover* $\tilde{X} \to X$ of X is an inverse system of étale covers $Y_i \to X$ such that for any étale cover $Y \to X$ there is a map $Y_i \to Y$ for i sufficiently large.

A *point* $\tilde{x}$ of the universal pro-étale cover $\tilde{X}$ is a compatible system of points $x_i \in Y_i$.

Let X be a smooth, geometrically connected, projective curve over a field k, $U \subset X$ an open subset with complement $D := X - U$, and $K := k(X) = k(U)$ the function field. Fix a universal pro-étale cover $\tilde{U} \to U$, and let $V_i \to U$ be the étale covers in the inverse system defining it. By Proposition 1.2.5, $V_i \to U$ extends uniquely to a finite morphism $Y_i \to X$, where Y_i is a smooth, geometrically connected, projective curve over k (note $Y_i \to X$ may be ramified over the points of D). This Y_i is in fact the normalisation of X in the function field of V_i (Theorem 1.2.4), which motivates the following definition.

Definition 1.4.13. With notation as in the above paragraph, the finite morphism $Y_i \to X$ is called the *normalisation of* X *in* V_i. The inverse system of these $Y_i \to X$ is called the *normalisation of* X *in* $\tilde{U}$, and is denoted $\tilde{X}_U \to X$.

A *point* $\tilde{x}$ of $\tilde{X}_U$ is a compatible system of points $x_i \in Y_i$.

Let $\bar{z} : \operatorname{Spec}\Omega \to U$ be a geometric point with image in U. By Lemma 1.4.5, the finite quotients of $\pi_1(U, \bar{z})$ are exactly the automorphism groups $\operatorname{Aut}(V|U)$, where $f : V \to U$ denotes an étale Galois cover of U (see Definition 1.4.1).

For each point $u \in U$ and each $v \in V$ with $f(v) = u$, the group $\operatorname{Aut}(V|U)$ contains, as in Definition 1.3.7, decomposition and inertia subgroups D_v, I_v. Indeed, by Theorem 1.4.10 we may assume that $\bar{z}$ has image the generic point of U. Then, since the morphism $\operatorname{Fib}_{\bar{z}}(V) = V_{\bar{z}} \to \operatorname{Spec} K$ is étale, $V_{\bar{z}}$ is isomorphic to the spectrum of a separable extension L of the function field K. Since the automorphism group $\operatorname{Aut}(V|U)$ acts freely and transitively on the geometric fibre $V_{\bar{z}}$, $L|K$ is a Galois extension, and $\operatorname{Aut}(V|U)$ contains decomposition and inertia subgroups D_v, I_v.

As above, Proposition 1.2.5 implies that every étale cover $f : V \to U$ extends uniquely to a finite morphism $f_Y : Y \to X$, where Y is a smooth, geometrically connected, projective curve over k (note f_Y may be ramified over the points of D). It also implies that every automorphism $\sigma \in \operatorname{Aut}(V|U)$ extends uniquely to an automorphism $\sigma_Y \in \operatorname{Aut}(Y|X)$. That is, $\operatorname{Aut}(V|U) = \operatorname{Aut}(Y|X)$, which means $\operatorname{Aut}(V|U)$ also contains decomposition and inertia subgroups associated to the points of D.

Thus, taking the inverse limit over the étale Galois covers of U, we may define decomposition and inertia subgroups of $\pi_1(U, \bar{z})$ associated to the points of U and of D.

Definition 1.4.14. With the above notation, let $x \in X$ be a point of X and $\tilde{x}$ a point above x in the normalisation $\tilde{X}_U$ of X in $\tilde{U}$. The *decomposition group* $D_{\tilde{x}}$ of $\tilde{x}$ is the stabiliser of $\tilde{x}$ under the action of $\pi_1(U, \bar{z})$. The *inertia group* $I_{\tilde{x}}$ is the kernel of the homomorphism $D_{\tilde{x}} \to \operatorname{Aut}(k(x)^{\mathrm{sep}}|k(x))$ (note the separable closure $k(x)^{\mathrm{sep}}$ is the compositum of the field extensions $k(y)|k(x)$ for all étale covers $Y \to X$, where y denotes the image in Y of an element of the inverse system of points defining $\tilde{x}$).

It follows immediately from the definition that decomposition and inertia groups for different choices of $\tilde{x}$ above x are conjugate, that is, for any $\sigma \in \pi_1(U, \bar{x})$ we have $I_{\sigma\tilde{x}} = \sigma I_{\tilde{x}} \sigma^{-1}$ and $D_{\sigma\tilde{x}} = \sigma D_{\tilde{x}} \sigma^{-1}$.

Lemma 1.4.15. *With the above notation, the sequence*

$$1 \longrightarrow I_{\tilde{x}} \longrightarrow D_{\tilde{x}} \longrightarrow G_{k(x)} \longrightarrow 1$$

is exact.

Proof. Take the inverse limit of the exact sequence in Lemma 1.3.8 over all étale Galois covers of U. The sequence stays exact in the inverse limit because the inverse system of finite cyclic inertia groups I_y satisfies the Mittag-Leffler condition. □

Proposition 1.4.16. *With the above notation, assume further that k has characteristic zero. If $x \in U$, the inertia group $I_{\tilde{x}}$ is trivial, while if $x \in D$ the inertia group $I_{\tilde{x}}$ is isomorphic as a $G_{k(x)}$-module to $\hat{\mathbb{Z}}(1) = \varprojlim_n \mu_n$, where the limit is taken over all positive integers n.*

Proof. Follows from Proposition 1.3.9 by taking the inverse limit over all étale Galois covers of U. □

The following Proposition is crucial in our study of ramification of sections in §3.4. See in particular Proposition 3.4.8 and Lemma 3.4.11.

Proposition 1.4.17. *With the above notation, assume further that k has characteristic zero and U is hyperbolic (see Definition 1.4.8). Then the decomposition and inertia groups of $\pi_1(U, \bar{z})$ satisfy the following properties.*

(i) *Inertia groups $I_{\tilde{x}}$ and $I_{\tilde{x}'}$ corresponding to distinct points $\tilde{x} \neq \tilde{x}'$ of $\tilde{X}_U$ intersect trivially.*

(ii) *For any point $\tilde{x}$ of $\tilde{X}_U$, the decomposition group $D_{\tilde{x}}$ is the normaliser of $I_{\tilde{x}}$ in $\pi_1(U, \bar{z})$.*

Proof. For the proof of (i) we refer to [HM11, Lemma 1.5]. For (ii), let $\sigma \in \pi_1(U, \bar{z})$, and suppose that σ normalises $I_{\tilde{x}}$. Then we have $I_{\tilde{x}} \cap I_{\sigma\tilde{x}} \neq 1$, thus by (i) we must have $\sigma\tilde{x} = \tilde{x}$ and hence $\sigma \in D_{\tilde{x}}$. Clearly any element of $D_{\tilde{x}}$ normalises $I_{\tilde{x}}$. □

Let $\varphi : X \to X'$ be a morphism of smooth, geometrically connected, projective curves over a field k, and let U' be an open subset of X' and $U := \varphi^{-1}(U')$ its preimage in X. Let $\tilde{U} \to U$ be a universal pro-étale cover of U and $\tilde{X}_U$ the normalisation of X in $\tilde{U}$, and similarly let $\tilde{U}' \to U'$ be a universal pro-étale cover of U' and $\tilde{X}'_{U'}$ the normalisation of X' in $\tilde{U}'$.

Let $(U_i \to U)_i$, respectively $(U'_j \to U')_j$ denote the inverse system of étale covers of U, resp. U' defining the universal pro-étale cover $\tilde{U}$, resp. $\tilde{U}'$. For each i, let Y_i denote the normalisation of X in U_i, and similarly, for each j, let Y'_j denote the normalisation of X' in U'_j.

Let x be a point of X and $\tilde{x}$ a point of $\tilde{X}_U$ above x, and let $(x_i)_i$ denote the compatible system of points defining $\tilde{x}$. For each j, $U'_j \times_{U'} U \to U$ is an étale cover of U, hence there is a morphism $U_i \to U'_j \times_{U'} U$ for i sufficiently large. Composing this with the projection $U'_j \times_{U'} U \to U'_j$, we therefore have a morphism $U_i \to U'_j$, which, by Proposition 1.2.5, extends uniquely to a morphism $Y_i \to Y'_j$. The images of the x_i via these morphisms define a point of $\tilde{X}'_{U'}$, which we will denote by $\varphi(\tilde{x})$.

Lemma 1.4.18. *With the above notation, let $\bar{z} : \operatorname{Spec}\Omega \to U$ and $\bar{z}' : \operatorname{Spec}\Omega \to U'$ be geometric points such that $\varphi \circ \bar{z} = \bar{z}'$, and let $\varphi_* : \pi_1(U, \bar{z}) \to \pi_1(U', \bar{z}')$ denote the homomorphism induced by $\varphi|_U$. Then the image of the decomposition subgroup $D_{\tilde{x}} \subset \pi_1(U, \bar{z})$ under φ_* is contained in $D_{\varphi(\tilde{x})} \subset \pi_1(U', \bar{z}')$.*

Proof. Since $D_{\tilde{x}}$ fixes the point $x_i \in Y_i$, it fixes the image of x_i in $Y'_j \times_{X'} X$, thus it fixes the image of x_i in Y'_j. Since this holds for every j, by definition of the homomorphism φ_* this means that the image of $D_{\tilde{x}}$ under φ_* fixes $\varphi(\tilde{x})$, that is, it is contained in $D_{\varphi(\tilde{x})}$. □

Let X be a smooth, connected, projective curve over a field k, $U \subseteq X$ an open subset, and $K := k(X) = k(U)$ the function field. The fundamental group of U has a realisation as the Galois group of an extension of K. Let $\bar{\xi} : \operatorname{Spec}\Omega \to U$ be a geometric point of U whose image is the generic point ξ of X. Then Ω necessarily contains an algebraic closure $\overline{K}$ of K.

Theorem 1.4.19. *With the notation of the above paragraph, there is a canonical isomorphism*

$$\pi_1(U, \bar{\xi}) \simeq \operatorname{Gal}(K_U|K)$$

where K_U denotes the maximal extension of K inside $\overline{K}$ which is unramified with respect to the valuations corresponding to the closed points of U.

In particular, there is a canonical isomorphism

$$\pi_1(X, \bar{\xi}) \simeq \operatorname{Gal}(K^{\mathrm{ur}}|K)$$

where K^{ur} denotes the maximal everywhere unramified extension of K inside $\overline{K}$.

Proof. The étale fundamental group $\pi_1(U, \bar{\xi})$ can be written as the projective limit of its finite quotients, which are exactly the automorphism groups $\mathrm{Aut}(V|U)$ with $f : V \to U$ étale Galois covers of U. As in the discussion before Definition 1.4.14, each such étale cover $f : V \to U$ extends uniquely to a finite morphism $f_Y : Y \to X$, where Y is a smooth, connected, projective curve over k, and we have $\mathrm{Aut}(V|U) = \mathrm{Aut}(Y|X)$. Thus we may write

$$\pi_1(U, \bar{\xi}) \simeq \varprojlim_{V \to U} \mathrm{Aut}(Y|X)$$

where the limit is taken over the étale Galois covers $V \to U$. By the equivalence of categories in Theorem 1.2.4, each finite morphism $Y \to X$ correponds to an extension of function fields $k(Y)|k(X)$ which is unramified with respect to the valuations ν_x with $x \in U$, and we may write

$$\pi_1(U, \bar{\xi}) \simeq \varprojlim_{L|K} \mathrm{Gal}(L|K) \simeq \mathrm{Gal}(K_U|K)$$

where the limit is taken over all finite extensions $L|K$ which are unramified with respect to the valuations ν_x with $x \in U$. $\square$

Definition 1.4.20. Let X be a smooth, connected, projective curve over a field k, $K := k(X)$ its function field, and $\overline{K}$ an algebraic closure of K. The *absolute Galois group of* X, denoted G_X, is the absolute Galois group of the function field of X

$$G_X := \mathrm{Gal}(K^{\mathrm{sep}}|K)$$

where K^{sep} is the separable closure of K inside $\overline{K}$.

A geometric point $\bar{\xi} : \operatorname{Spec} \Omega \to X$ with image the generic point of X naturally determines a choice of separable closure of K, and may be considered a "base point" of G_X in the following sense. For each open subset $U \subset X$, $\bar{\xi} : \operatorname{Spec} \Omega \to X$ determines a geometric point $\bar{\xi} : \operatorname{Spec} \Omega \to U$ with image the generic point of U, hence a base point of the étale fundamental group of U. Thus we can take the projective limit of the $\pi_1(U, \bar{\xi})$ as U ranges over all the open subsets of X, giving the following useful expression for the absolute Galois group of X.

Lemma 1.4.21. *With X, K and $\bar{\xi}$ as above, there is a canonical isomorphism*

$$G_X \simeq \varprojlim_{U \subset X \text{ open}} \pi_1(U, \bar{\xi})$$

where the open subsets $U \subset X$ are partially ordered by inclusion.

Proof. By Theorem 1.4.19, we have:

$$\varprojlim_{U \subset X \text{ open}} \pi_1(U, \bar{\xi}) \simeq \varprojlim_{U \subset X \text{ open}} \mathrm{Gal}(K_U|K) \simeq \mathrm{Gal}(K^{\mathrm{sep}}|K)$$

where K^{sep} is the separable closure of K inside Ω. □

Definition 1.4.22. With the notation of Definition 1.4.20, let $x \in X$ be a closed point of X and $\tilde{x}$ a valuation of K^{sep} extending the valuation ν_x of K corresponding to x. We will refer to $\tilde{x}$ as an *extension of x to K^{sep}*.

The *decomposition group* $D_{\tilde{x}}$ of $\tilde{x}$ is the stabiliser of $\tilde{x}$ under the action of G_X. The *inertia group* $I_{\tilde{x}}$ is the kernel of the natural homomorphism $D_{\tilde{x}} \to G_{k(x)}$.

In light of Lemma 1.4.21, the decomposition and inertia subgroups of G_X can be interpreted as limits of the decomposition and inertia subgroups of the $\pi_1(U, \bar{\xi})$ as U ranges over all the open subsets of X. Indeed, a valuation $\tilde{x}$ of K^{sep} extending ν_x determines a compatible system of points $\tilde{x}_U$, parameterised by all the open subsets $U \subset X$, where each $\tilde{x}_U$ is a point above x in the normalisation $\tilde{X}_U$ of X in some universal pro-étale cover $\tilde{U}$ of U. This is because, for any open subset $U \subset X$, any étale cover $V \to U$ extends uniquely to a finite, separable morphism $Y \to X$, and the valuation $\tilde{x}$ naturally determines a valuation of the function field $k(Y)$ extending ν_x, that is, a point of Y above x. In this way, as V ranges over the étale covers of U in the inverse system defining a chosen universal pro-étale cover $\tilde{U} \to U$, $\tilde{x}$ determines a point of $\tilde{X}_U$.

By Lemma 1.4.21, the decomposition subgroup $D_{\tilde{x}} \subset G_X$ is then the inverse limit

$$D_{\tilde{x}} = \varprojlim_{U \subset X \text{ open}} D_{\tilde{x}_U}$$

of the decomposition subgroups $D_{\tilde{x}_U} \subset \pi_1(U, \bar{\xi})$. Thus Proposition 1.4.16 immediately implies the following.

Proposition 1.4.23. *For any closed point $x \in X$ and any extension $\tilde{x}$ of x to K^{sep}, the inertia subgroup $I_{\tilde{x}} \subset G_X$ is canonically isomorphic to $\hat{\mathbb{Z}}(1)$.*

1.5 The fundamental exact sequence and sections

Let X be a quasi-compact, geometrically integral, locally Noetherian scheme over a field k, $\bar{z} : \operatorname{Spec}\Omega \to X$ a geometric point, and $\bar{k}$ the algebraic closure of k in Ω. The composite morphism $\operatorname{Spec}\Omega \to X \to \operatorname{Spec} k$ determines a geometric point of the scheme $\operatorname{Spec} k$. Denoting $X_{\bar{k}} := X \times_k \bar{k}$ (see Notation), the geometric point $\bar{z}$ naturally defines a geometric point $\bar{z} : \operatorname{Spec}\Omega \to X_{\bar{k}}$, which we also denote by $\bar{z}$. Thus the projection and structural morphisms $X_{\bar{k}} \to X \to \operatorname{Spec} k$ give rise, by functoriality of the fundamental group, to homomorphisms $\pi_1(X_{\bar{k}}, \bar{z}) \to \pi_1(X, \bar{z}) \to G_k$.

Theorem 1.5.1. *With the above notation, the sequence*

$$1 \longrightarrow \pi_1(X_{\bar{k}}, \bar{z}) \longrightarrow \pi_1(X, \bar{z}) \longrightarrow G_k \longrightarrow 1$$

is exact.

Proof. See [Sza09, Proposition 5.6.1] or [GR71, Exposé IX, Théorème 6.1]. □

We shall refer to this as the *fundamental exact sequence* of X. The fundamental group $\pi_1(X_{\bar{k}}, \bar{z})$ will be called the *geometric fundamental group* of X.

The statement of the Theorem is easy to see in the following special case. Suppose that k is perfect, and let X be a smooth, geometrically connected, projective curve over k. Let $K := k(X)$ be the function field of X, $U \subset X$ an open subset, $\bar{z} : \operatorname{Spec}\Omega \to U$ a geometric point with image in U, and $\bar{k}$ the algebraic closure of k in Ω. Then the fundamental exact sequence

$$1 \longrightarrow \pi_1(U_{\bar{k}}, \bar{z}) \longrightarrow \pi_1(U, \bar{z}) \longrightarrow G_k \longrightarrow 1$$

is, by Theorem 1.4.19, the sequence of Galois groups

$$1 \longrightarrow \operatorname{Gal}(K_U | K\bar{k}) \longrightarrow \operatorname{Gal}(K_U | K) \longrightarrow G_k \longrightarrow 1$$

which is evidently exact.

Corollary 1.5.2. *Let X be a smooth, geometrically connected, projective curve over* *a field k. Then there is an exact sequence of absolute Galois groups*

$$1 \longrightarrow G_{X_{\bar{k}}} \longrightarrow G_X \longrightarrow G_k \longrightarrow 1$$

Proof. Let $\bar{\xi} : \operatorname{Spec} \Omega \to X$ be a geometric point with image the generic point of X. By Lemma 1.4.21, taking the inverse limit of the fundamental exact sequences

$$1 \longrightarrow \pi_1(U_{\bar{k}}, \bar{\xi}) \longrightarrow \pi_1(U, \bar{\xi}) \longrightarrow G_k \longrightarrow 1$$

over the open subsets $U \subset X$ gives the required exact sequence (the sequence stays exact in the inverse limit by [RZ10, Proposition 2.2.4]). □

We now investigate sections of the above exact sequences. We recall some general definitions from cohomology of profinite groups, which we state without assuming any of our groups are abelian. See [Ser97, Ch. 1, §5] for an account of non-abelian cohomology of profinite groups, and [NSW08, Ch. 1] for a thorough exposition of abelian cohomology of profinite groups.

Let G be a profinite group and A a (not necessarily abelian) group with a continuous action of G (with respect to the discrete topology on A). We denote the action of $\sigma \in G$ on $a \in A$ by σa. A *cocycle* of G is a continuous function $a : G \to A$ such that

$$a(\sigma\tau) = a(\sigma) \cdot \sigma a(\tau)$$

for any $\sigma, \tau \in G$. Two cocycles a, a' of G are *cohomologous* if there exists $b \in A$ such that $a'(\sigma) = b\, a(\sigma)\, \sigma b^{-1}$ for every $\sigma \in G$. This is an equivalence relation on the set of cocycles of G, and the set of equivalence classes is called the *first cohomology set of G in A*, denoted $H^1(G, A)$. This set has a distinguished element, namely the cohomology class of the trivial cocycle $\sigma \mapsto 1\, \forall \sigma \in G$, giving it the structure of a "pointed set". When A is abelian, $H^1(G, A)$ is a group, namely the first cohomology group of G in A.

Let E be an extension of G by A, that is, a group which sits in an exact sequence

$$1 \longrightarrow A \longrightarrow E \xrightarrow{r} G \longrightarrow 1$$

Definition 1.5.3. A *section* of the above exact sequence is a continuous group homomorphism $s : G \to E$ such that $r \circ s = \mathrm{Id}_G$. Such a section may also be called a *section of* E if the exact sequence is understood.

Two sections s, s' of the sequence are *conjugate* if there exists $a \in A$ such that $s'(\sigma) = as(\sigma)a^{-1}$ for all $\sigma \in G$. We denote by $[s]$ the conjugacy class of a section s, and by $\overline{\mathrm{Sec}}_E$ the set of conjugacy classes of sections of E.

The *semi-direct product* $A \rtimes G$ is the group $\{(a, \sigma) | a \in A, \sigma \in G\}$ with multiplication $(a, \sigma) \cdot (a', \sigma') = (a\,\sigma a', \sigma\sigma')$. There is a natural exact sequence

$$1 \longrightarrow A \longrightarrow A \rtimes G \xrightarrow{r} G \longrightarrow 1$$

The inclusion $G \hookrightarrow A \rtimes G$, $\sigma \mapsto (1, \sigma)$ obviously defines a section of this sequence, which we shall refer to as the *trivial section*. The set $\overline{\mathrm{Sec}}_{A \rtimes G}$ has the structure of a pointed set, with distinguished element the conjugacy class of the trivial section.

Proposition 1.5.4. *Let* $1 \longrightarrow A \longrightarrow E \xrightarrow{r} G \longrightarrow 1$ *be an exact sequence of profinite groups. If there exists a section* $s_0 : G \to E$ *of this sequence, then there is a canonical isomorphism* $E \simeq A \rtimes G$ *defined by:*

$$\begin{aligned} E &\xrightarrow{\sim} A \rtimes G \\ e &\longmapsto (e \cdot s_0(r(e)^{-1}), r(e)) \\ a \cdot s_0(\sigma) &\longleftarrow\!\!\shortmid (a, \sigma) \end{aligned}$$

Note that a bijection of pointed sets means that it takes the distinguished element of one set to that of the other.

Proof. This statement may be found in [Mac63, Ch. IV, §3], though with the unnecessary assumption that A is abelian. □

Theorem 1.5.5. *Let* $1 \longrightarrow A \longrightarrow E \xrightarrow{r} G \longrightarrow 1$ *be an exact sequence of profinite groups. Then either:*

(i) *the sequence has no sections; or*

(ii) *the sequence has a section* $s_0 : G \to E$*, and there is a canonical bijection of pointed sets*

$$H^1(G, A) \simeq \overline{\mathrm{Sec}}_E$$

where the distinguished element of $\overline{\mathrm{Sec}}_E$ *is the conjugacy class of the section* s_0.

Proof of Theorem 1.5.5. To prove the Theorem we need to show that if E has a section then there is a canonical bijection of pointed sets $H^1(G,A) \simeq \overline{\mathrm{Sec}}_E$. If there exists a section $s_0 : G \to E$ then, under the canonical isomorphism of Proposition 1.5.4, it corresponds to the trivial section of $A \rtimes G$. Indeed, we have $s_0(\sigma) \mapsto (s_0(\sigma) \cdot s(r \circ s_0(\sigma)^{-1}), r \circ s_0(\sigma)) = (s_0(\sigma) \cdot s_0(\sigma^{-1}), \sigma) = (1, \sigma)$. Thus the isomorphism $E \simeq A \rtimes G$ induces a bijection of pointed sets $\overline{\mathrm{Sec}}_E \simeq \overline{\mathrm{Sec}}_{A \rtimes G}$, where the conjugacy class of the section s_0 maps to that of the trivial section of $A \rtimes G$.

It remains to show that there exists a canonical bijection of pointed sets

$$H^1(G,A) \simeq \overline{\mathrm{Sec}}_{A \rtimes G}$$

For a cocycle $a : G \to A$, one can define a section $s_a : G \to A \rtimes G$ by $s_a(\sigma) = (a(\sigma), \sigma)$. Conversely, any section $s : G \to A \rtimes G$ must be of this form - indeed, $s(\sigma)$, for some $\sigma \in G$, has a unique presentation in the form $s(\sigma) = (a_\sigma, \sigma)$. For $\sigma, \sigma' \in G$ we therefore have $s(\sigma\sigma') = s(\sigma)s(\sigma') = (a_\sigma, \sigma)(a_{\sigma'}, \sigma') = (a_\sigma \, \sigma a_{\sigma'}, \sigma\sigma')$. The function $a : G \to A$, $\sigma \mapsto a_\sigma$ is evidently a cocycle. Moreover, a section conjugate to s corresponds to a cocycle cohomologous to a_σ. Indeed, for any $b \in A$ we have $bs(\sigma)b^{-1} = (b,1) \cdot (a_\sigma, \sigma) \cdot (b^{-1}, 1) = (b\, a_\sigma, \sigma) \cdot (b^{-1}, 1) = (b\, a_\sigma \, \sigma b^{-1}, \sigma)$.

The map $a \mapsto s_a$ therefore defines a bijection $H^1(G,A) \simeq \overline{\mathrm{Sec}}_{A \rtimes G}$, and it is a bijection of pointed sets since the conjugacy class of the trivial cocycle clearly maps to that of the trivial section. The composite

$$H^1(G,A) \simeq \overline{\mathrm{Sec}}_{A \rtimes G} \simeq \overline{\mathrm{Sec}}_E$$

is the required bijection of pointed sets. $\square$

For the remainder of this section, X will denote a smooth, geometrically connected, projective curve over a field k, U an open subset of X, $D := X - U$ its complement, and $\bar{z} : \mathrm{Spec}\,\Omega \to U$ a geometric point with image in U. Let us also fix a universal pro-étale cover $\tilde{U} \to U$ (Definition 1.4.12).

Let x be a k-rational point of U, which is the image of a morphism $\mathrm{Spec}\,k \to U$ (that is, an element of $U(k)$). This morphism gives rise, by functoriality of the fundamental group, to sections of the fundamental exact sequence of U.

Proposition 1.5.6. *With the above notation, the k-rational point $x \in U(k)$ determines a conjugacy class of sections $s_x : G_k \to \pi_1(U, \bar{z})$ of the fundamental exact* *sequence of U, where the image of a section in this class is contained in a decomposition group $D_{\tilde{x}}$ for some point $\tilde{x}$ in $\tilde{U}$ above x.*

Proof. The section $\operatorname{Spec} k \to U$ defining x gives rise, by functoriality of the fundamental group, to a section $s'_x : G_k \to \pi_1(U, \bar{x})$, where $\bar{x} : \operatorname{Spec} \Omega' \to U$ is a geometric point with image x. Note that composing s'_x with conjugation by $\pi_1(U_{\bar{k}}, \bar{x})$ determines another section of $\pi_1(U, \bar{x})$.

By Theorem 1.4.10, there is an isomorphism $\pi_1(U, \bar{x}) \simeq \pi_1(U, \bar{z})$, which is uniquely determined up to conjugation by $\pi_1(U, \bar{z})$. Thus the composition of s'_x with this isomorphism yields a section $s_x : G_k \to \pi_1(U, \bar{z})$ which is uniquely determined up to conjugation by $\pi_1(U_{\bar{k}}, \bar{z})$. (It must be conjugation by $\pi_1(U_{\bar{k}}, \bar{z})$ so that s_x induces the identity map on G_k.)

$$\begin{tikzcd}
 & & \pi_1(U, \bar{x}) \arrow[d, "\wr"] \arrow[dr] & & \\
1 \arrow[r] & \pi_1(U_{\bar{k}}, \bar{z}) \arrow[r] & \pi_1(U, \bar{z}) \arrow[r] & G_k \arrow[r] \arrow[ul, bend right, "s'_x"'] \arrow[l, bend left, "s_x"] & 1
\end{tikzcd}$$

To show that the image of s_x is contained in a decomposition group $D_{\tilde{x}}$, let $f : V \to U$ be an étale cover. Then the morphism $V_x = V \times_U \operatorname{Spec} k \to \operatorname{Spec} k$ is also étale. Thus V_x is isomorphic to a disjoint union $\coprod_i \operatorname{Spec} l_i$, where each l_i is a separable extension of k, and an element $\sigma \in G_k$ acts on V_x by acting on one of the components $\operatorname{Spec} l_i$. In other words, G_k fixes a point in the fibre above x. Taking the limit over the étale covers of U, this means $s'_x(\sigma) \in D_{\tilde{x}} \subset \pi_1(U, \bar{x})$ for some $\tilde{x}$ in $\tilde{U}$ above x. Via the isomorphism in the diagram, we therefore have $s_x(G_k) \subseteq D_{\tilde{x}} \subset \pi_1(U, \bar{z})$. $\square$

Let x be any closed point of X, and let $\tilde{x}$ be a point above x in the normalisation $\tilde{X}_U$ of X in $\tilde{U}$ (Definition 1.4.13). Consider again the exact sequence of Lemma 1.4.15, which injects into the fundamental exact sequence:

$$\begin{tikzcd}
1 \arrow[r] & I_{\tilde{x}} \arrow[r] \arrow[d, hook] & D_{\tilde{x}} \arrow[r] \arrow[d, hook] & G_{k(x)} \arrow[r] \arrow[d, hook] & 1 \\
1 \arrow[r] & \pi_1(U_{\bar{k}}, \bar{z}) \arrow[r] & \pi_1(U, \bar{z}) \arrow[r] & G_k \arrow[r] & 1
\end{tikzcd} \tag{1.3}$$

When $x \in X(k)$ is a k-rational point, so that $G_{k(x)} = G_k$, a splitting of the upper exact sequence naturally defines one of the fundamental exact sequence.

Definition 1.5.7. A section $s : G_k \to \pi_1(U, \bar{z})$ is called *cuspidal* if it factors through $D_{\tilde{x}}$ for some (necessarily k-rational) point $x \in D$ and some $\tilde{x} \in \tilde{X}_U$ above x.

Suppose now that k has characteristic zero. In this case, when $x \in U$ the inertia group $I_{\tilde{x}}$ is trivial (Proposition 1.4.16), hence we have an isomorphism $D_{\tilde{x}} \simeq G_{k(x)}$ and the upper exact sequence in diagram 1.3 obviously splits. If x is a k-rational point of U then $D_{\tilde{x}} \simeq G_k$, so x induces a section $s_x : G_k \to \pi_1(U, \bar{z})$ of the fundamental exact sequence with image $D_{\tilde{x}}$. Such a section lies in the conjugacy class of sections induced by functoriality as discussed above. In particular, the image of a section induced by functoriality from a k-rational point of U not only has image *contained in* a decomposition group $D_{\tilde{x}}$, it has image *exactly* $D_{\tilde{x}}$, i.e. $s_x(G_k) = D_{\tilde{x}}$.

But the exact sequence also splits when x is contained in the complement $D = X - U$.

Proposition 1.5.8. *With the above notation, assume that k has characteristic zero, and let x be a closed point in D and $\tilde{x}$ a point of $\tilde{X}_U$ above x. Then the exact sequence $1 \to I_{\tilde{x}} \to D_{\tilde{x}} \to G_{k(x)} \to 1$ splits.*

Proof. For ease of notation, let us write $l := k(x)$. By Proposition 1.4.16, the inertia group $I_{\tilde{x}}$ is isomorphic to $\hat{\mathbb{Z}}(1)$ (recall that k is assumed to have characteristic zero). Moreover, denoting by $l((t))$ the field of formal Laurent series with coefficients in l, the decomposition group $D_{\tilde{x}}$ is isomorphic to the absolute Galois group $\mathrm{Gal}(\overline{l((t))}|l((t)))$, and G_l is isomorphic to the Galois group $\mathrm{Gal}(l((t))^{\mathrm{ur}}|l((t)))$ of the maximal unramified extension of $l((t))$ (see [Ser79, Ch. II, §3, Corollary 4 and Ch. II, §4, Theorem 2]). Thus to find a section of the above exact sequence is to find an extension $L|l((t))$ whose Galois group $\mathrm{Gal}(L|l((t)))$ is isomorphic to $\hat{\mathbb{Z}}(1)$. Indeed, such an extension is totally ramified and so we have an isomorphism $\mathrm{Gal}(\overline{l((t))}|L) \simeq G_l$, which defines a section of $D_{\tilde{x}}$.

Such an extension L may be obtained as follows: choose a compatible system of n-th roots t_n of t as n ranges over the positive integers (so $(t_{nn'})^{n'} = t_n$), and let L_n denote the extension $l((t))(t_n)$ of $l((t))$ obtained by adjoining the n-th root t_n. The inverse limit $L := \varprojlim_n L_n$ defines the required extension of $l((t))$ with $\mathrm{Gal}(L|l((t))) \simeq \hat{\mathbb{Z}}(1)$. □

Thus there exist cuspidal sections of $\pi_1(U, \bar{z})$ when k has characteristic zero.

Definition 1.5.9. With the above notation, assume that k has characteristic zero, and let x be a k-rational point of D and $\tilde{x}$ a point of $\tilde{X}_U$ above x. The set of $I_{\tilde{x}}$-conjugacy classes of sections of the sequence $1 \to I_{\tilde{x}} \to D_{\tilde{x}} \to G_k \to 1$ is called the *packet* of cuspidal sections associated to $\tilde{x}$.

By Proposition 1.5.8 and Theorem 1.5.5, for a k-rational point x of D and a point $\tilde{x} \in \tilde{X}_U$ above x, the packet of cuspidal sections associated to $\tilde{x}$ is in bijection with the cohomology group $H^1(G_k, I_{\tilde{x}})$. Note that this is the set of conjugacy classes with respect to conjugation by the inertia subgroup $I_{\tilde{x}} \subset \pi_1(U_{\bar{k}}, \bar{z})$. Conjugation by an arbitrary element $\sigma \in \pi_1(U_{\bar{k}}, \bar{z})$ induces a map of packets

$$\begin{aligned} H^1(G_k, I_{\tilde{x}}) &\xrightarrow{\sigma} H^1(G_k, I_{\sigma\tilde{x}}) \\ s &\longmapsto \sigma s \sigma^{-1} \end{aligned}$$

Since x is k-rational, as σ ranges over the elements of $\pi_1(U_{\bar{k}}, \bar{z})$ the points $\sigma\tilde{x}$ range over all the points in $\tilde{X}_U$ above x. Thus, associated to a k-rational point x is a set of packets $\{H^1(G_k, I_{\tilde{x}})\}_{\tilde{x}}$ enumerated by the points $\tilde{x}$ of $\tilde{X}_U$ above x. To say that a cuspidal section s is "associated to x" is to say that s belongs to a class in one of the packets $H^1(G_k, I_{\tilde{x}})$ for some $\tilde{x} \in \tilde{X}_U$ above x.

Two cuspidal sections s, s' which are associated to the same k-rational point x need not be conjugate by an element of $\pi_1(U_{\bar{k}}, \bar{z})$. Indeed, there will exist some $\sigma \in \pi_1(U_{\bar{k}}, \bar{z})$ such that, via the above map, s and $\sigma s' \sigma^{-1}$ belong to the same packet $H^1(G_k, I_{\tilde{x}})$, but their $I_{\tilde{x}}$-conjugacy classes need not coincide. This contrasts with the situation for sections arising by functoriality from k-rational points of U: any two sections arising by functoriality from the same $x \in U(k)$ are necessarily conjugate by some element of $\pi_1(U_{\bar{k}}, \bar{z})$ (see Proposition 1.5.6). The difference is down to the structure of $I_{\tilde{x}}$ (see Proposition 1.4.16): this inertia group is trivial when $x \in U(k)$, but when $x \in D(k)$ it is isomorphic to $\hat{\mathbb{Z}}(1)$ and thus gives rise to a non-trivial packet $H^1(G_k, I_{\tilde{x}}) \simeq H^1(G_k, \hat{\mathbb{Z}}(1))$.

Let us now fix a choice of geometric point $\bar{\xi} : \operatorname{Spec} \Omega \to X$ with image the generic point of X. Let $K := k(X)$ denote the function field of X and K^{sep} the separable closure of K in Ω. Consider again the exact sequence of absolute Galois groups from Corollary 1.5.2.

$$1 \longrightarrow G_{X_{\bar{k}}} \longrightarrow G_X \longrightarrow G_k \longrightarrow 1$$

Let x be a closed point of X, and let $\tilde{x}$ be an extension of x to K^{sep} (see Definition 1.4.22). Taking the inverse limit of diagram 1.3 as U ranges over all the open subsets of X, with $\bar{\xi}$ in place of the base point $\bar{z}$, we obtain a commutative diagram with exact rows:

$$\begin{tikzcd} 1 \arrow[r] & I_{\tilde{x}} \arrow[r] \arrow[d, hook] & D_{\tilde{x}} \arrow[r] \arrow[d, hook] & G_{k(x)} \arrow[r] \arrow[d, hook] & 1 \\ 1 \arrow[r] & G_{X_{\bar{k}}} \arrow[r] & G_X \arrow[r] & G_k \arrow[r] & 1 \end{tikzcd}$$

(See [RZ10, Proposition 2.2.4] for the statement that the horizontal exact sequences in diagram 1.3 stay exact in the inverse limit.) When $x \in X(k)$ is a k-rational point, a splitting of the upper exact sequence naturally defines a section of G_X. The following Lemma follows exactly as in Proposition 1.5.8.

Lemma 1.5.10. *With the above notation, assume further that k has characteristic zero. Then a k-rational point x of X induces, for each extension $\tilde{x}$ of x to K^{sep}, a set of $I_{\tilde{x}}$-conjugacy classes of sections of G_X isomorphic to $H^1(G_k, I_{\tilde{x}}) \simeq H^1(G_k, \hat{\mathbb{Z}}(1))$. The image of such a section is contained in the decomposition subgroup $D_{\tilde{x}} \subset G_X$.*

Definition 1.5.11. With the notation and assumptions of Lemma 1.5.10, the set of $I_{\tilde{x}}$-conjugacy classes of sections of G_X is called the *packet* of sections of G_X associated to $\tilde{x}$.

By Lemma 1.4.21, for any open subset $U \subset X$, the fundamental group $\pi_1(U, \bar{\xi})$ is naturally a quotient of G_X. In fact we have a commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & G_{X_{\bar{k}}} \arrow[r] \arrow[d, two heads] & G_X \arrow[r] \arrow[d, two heads] & G_k \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & \pi_1(U_{\bar{k}}, \bar{\xi}) \arrow[r] & \pi_1(U, \bar{\xi}) \arrow[r] & G_k \arrow[r] & 1 \end{tikzcd}$$

via which a section $s : G_k \to G_X$ naturally induces a section $s_U : G_k \to \pi_1(U, \bar{\xi})$. This follows simply by commutativity, which implies that the kernel I_U of the projection $G_X \twoheadrightarrow \pi_1(U, \bar{\xi})$ is contained in the kernel of the projection $G_X \twoheadrightarrow G_k$, hence the image of the section s is disjoint from I_U. Note that, by Theorem 1.4.19, this kernel I_U is the group $\langle I_{\tilde{x}} \rangle$ normally generated by the inertia groups $I_{\tilde{x}}$ for all closed points $x \in U$ and all $\tilde{x}$ in some fixed universal pro-étale cover $\tilde{U}$ of U.

Thus, by Lemma 1.4.21, a section s of G_X determines, and is determined by, a compatible system of sections $s_U : G_k \to \pi_1(U, \bar{\xi})$ for the open subsets $U \subset X$.

When k has characteristic zero, sections of G_X induced as in Lemma 1.5.10 by k-rational points of X may be considered "cuspidal" in the following sense. Let $s : G_k \to G_X$ be a section arising from a k-rational point $x \in X(k)$, and let $\big(s_U : G_k \to \pi_1(U,\bar{\xi})\big)_{U\subset X \text{ open}}$ be the corresponding compatible system. If x is contained in some open subset $U \subset X$ then it gives rise to a single conjugacy class of sections of $\pi_1(U,\bar{\xi})$, and s_U is an element of this class. Denoting $U' = U - \{x\}$, the point x gives rise to a set of packets of cuspidal sections of $\pi_1(U',\bar{\xi})$, and the section $s_{U'}$ is an element of a class in one of these packets. Intuitively, when we delete x from U the conjugacy class of sections of $\pi_1(U,\bar{\xi})$ induced by x "becomes" a set of packets of cuspidal sections, adding a copy of $\hat{\mathbb{Z}}(1)$ into the decomposition subgroup $D_{\tilde{x}} \subset \pi_1(U,\bar{\xi})$. When we take the projective limit $\varprojlim_U \pi_1(U,\bar{\xi})$ we are making U smaller and smaller, successively deleting points from U, so that all the sections s_U "become" cuspidal.

1.6 Geometrically abelian fundamental groups

In this section we will deal with proper schemes over a field, and consider étale fundamental groups and their quotients. Let X be a proper, integral, normal, locally Noetherian scheme over a field k, and $\bar{z} : \operatorname{Spec}\Omega \to X$ a geometric point.

Definition 1.6.1. Let $\mathcal{N}$ denote the set of open normal subgroups $N \lhd \pi_1(X_{\bar{k}},\bar{z})$ such that $\pi_1(X_{\bar{k}},\bar{z})/N$ is abelian. The *maximal abelian quotient* $\pi_1(X_{\bar{k}},\bar{z})^{\mathrm{ab}}$ of the geometric fundamental group is the quotient

$$\pi_1(X_{\bar{k}},\bar{z})^{\mathrm{ab}} := \pi_1(X_{\bar{k}},\bar{z}) / \bigcap_{N\in\mathcal{N}} N$$

We consider each $N \in \mathcal{N}$ a subgroup of the fundamental group $\pi_1(X,\bar{z})$ via the injection $\pi_1(X_{\bar{k}},\bar{z}) \hookrightarrow \pi_1(X,\bar{z})$. Then the intersection $\cap_{N\in\mathcal{N}} N$ is a normal subgroup of $\pi_1(X,\bar{z})$, since it is a characteristic subgroup of $\pi_1(X_{\bar{k}},\bar{z})$ (indeed, for each $N \in \mathcal{N}$, any characteristic subgroup of N is a normal subgroup of $\pi_1(X_{\bar{k}},\bar{z})$). The *geometrically abelian quotient* of the fundamental group $\pi_1(X,\bar{z})$ is the quotient

$$\pi_1(X,\bar{z})^{(\mathrm{ab})} := \pi_1(X,\bar{z}) / \bigcap_{N\in\mathcal{N}} N$$

Thus the geometrically abelian quotient is defined by the pushout diagram

$$\begin{tikzcd} 1 \arrow[r] & \pi_1(X_{\bar{k}}, \bar{z}) \arrow[r] \arrow[d, two heads] & \pi_1(X, \bar{z}) \arrow[r] \arrow[d, two heads] & G_k \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & \pi_1(X_{\bar{k}}, \bar{z})^{\mathrm{ab}} \arrow[r] & \pi_1(X, \bar{z})^{(\mathrm{ab})} \arrow[r] & G_k \arrow[r] & 1 \end{tikzcd} \tag{1.4}$$

where the upper row is the fundamental exact sequence.

Remark 1.6.2. By Theorem 1.4.10, for different choices of base point $\bar{z}$, $\bar{z}'$ there exists a *unique* isomorphism $\pi_1(X_{\bar{k}}, \bar{z})^{\mathrm{ab}} \simeq \pi_1(X_{\bar{k}}, \bar{z}')^{\mathrm{ab}}$. Thus the geometrically abelian quotient of the fundamental group is not dependent on the choice of base point, so we could omit it from our notation. However, we will include the base point to be consistent with the other fundamental groups in the above diagram.

Now suppose that X is a smooth, geometrically connected, projective curve over k. Assume that $X(k) \neq \emptyset$, and fix a k-rational point $z_0 \in X(k)$. Let J denote the Jacobian of X (see Definition 1.1.9).

Lemma 1.6.3. *There is a unique morphism $\iota : X \to J$ taking z_0 to 0. Moreover, ι is a closed immersion.*

Proof. See the paragraph following the proof of Proposition 2.1 in [Mil86] for the first statement, and Proposition 2.3 in loc. cit. for the second statement. □

For any rational point $x \in X(k)$, the map $\iota|_{X(k)} : X(k) \hookrightarrow J(k) = \mathrm{Pic}^0(X)$ takes x to the class of the divisor $[x] - [z_0]$ (note this divisor has degree zero, thus is indeed an element of $\mathrm{Pic}^0(X)$ - see Definition 1.1.5).

Let $\bar{z}_0 : \operatorname{Spec} \Omega \to X$ be a geometric point of X with image z_0, and $\bar{0} := \iota \circ \bar{z}_0 : \operatorname{Spec} \Omega \to J$ the induced geometric point of J with image 0. Then the morphism ι induces, by functoriality of the fundamental group, a canonical homomorphism

$$\iota_* : \pi_1(X, \bar{z}_0) \to \pi_1(J, \bar{0})$$

and similarly a canonical homomorphism of geometric fundamental groups

$$\bar{\iota}_* : \pi_1(X_{\bar{k}}, \bar{z}_0) \to \pi_1(J_{\bar{k}}, \bar{0})$$

Lemma 1.6.4. *There is an isomorphism $\pi_1(X_{\bar{k}}, \bar{z}_0)^{\mathrm{ab}} \simeq \pi_1(J_{\bar{k}}, \bar{0})$. In particular, the homomorphism $\bar{\iota}_*$ factors through the maximal abelian quotient $\pi_1(X_{\bar{k}}, \bar{z}_0)^{\mathrm{ab}}$.*

Proof. See [Mil86, Proposition 9.1] □

The morphism ι induces the following diagram of exact sequences.

$$\begin{tikzcd} 1 \arrow[r] & \pi_1(X_{\bar{k}}, \bar{z}_0)^{\mathrm{ab}} \arrow[r] \arrow[d, "\bar{\iota}_*"', "\wr"] & \pi_1(X, \bar{z}_0)^{(\mathrm{ab})} \arrow[r] \arrow[d, "\iota_*"'] & G_k \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & \pi_1(J_{\bar{k}}, \bar{0}) \arrow[r] & \pi_1(J, \bar{0}) \arrow[r] & G_k \arrow[r] & 1 \end{tikzcd} \tag{1.5}$$

Therefore, since $\bar{\iota}_*$ is an isomorphism, so is ι_*.

We can also identify the maximal abelian quotient $\pi_1(X_{\bar{k}}, \bar{z}_0)^{\mathrm{ab}}$ with the *Tate module* of the Jacobian $J_{\bar{k}}$, which is defined as follows.

Definition 1.6.5. Let A be an abelian group. For any two positive integers M, N such that $N|M$, we have a "multiplication by M/N" map $\phi_{MN} : A[M] \to A[N]$, $a \mapsto (M/N) \cdot a$, where $A[M]$, respectively $A[N]$ is the group of M-, resp. N-torsion elements of A (see Notation). Then $(A[N], \phi_{MN})$ defines an inverse system, and its inverse limit is called the *Tate module* of A, denoted TA:

$$TA := \varprojlim_N A[N]$$

In particular, if ℓ is a prime integer, for every positive integer n we have a "multiplication by ℓ" map $\phi_\ell : A[\ell^{n+1}] \to A[\ell^n]$, via which we define the *ℓ-adic Tate module* $T_\ell A$:

$$T_\ell A := \varprojlim_n A[\ell^n]$$

In particular we may consider the Tate module of the abelian group $J_{\bar{k}}(\bar{k})$. We abbreviate by writing $TJ := TJ_{\bar{k}}(\bar{k}) \simeq TJ(\bar{k})$ (see Notation).

Lemma 1.6.6. *There is an isomorphism of G_k-modules $\pi_1(J_{\bar{k}}, \bar{0}) \simeq TJ$.*

Proof. See [Sza09, Theorem 5.6.10]. □

We now investigate sections of the geometrically abelian quotient of the fundamental group. Let s_{z_0} denote the section of $\pi_1(X, \bar{z}_0)$ induced from the k-rational point z_0 by functoriality as in Proposition 1.5.6. By commutativity of diagram 1.4, s_{z_0} induces a section $s_{z_0}^{\mathrm{ab}} : G_k \to \pi_1(X, \bar{z}_0)^{(\mathrm{ab})}$ of the geometrically abelian quotient. Thus the group $\pi_1(X, \bar{z}_0)^{(\mathrm{ab})}$ has a section, so by Theorem 1.5.5 we may identify the set of conjugacy classes of sections of $\pi_1(X, \bar{z}_0)^{(\mathrm{ab})}$ with the Galois cohomology group $H^1(G_k, TJ)$, the section s_{z_0} corresponding to the conjugacy class of the trivial section.

Any other k-rational point $x \in X(k)$ induces, by functoriality, a section $s_x : G_k \to \pi_1(X, \bar{z}_0)$, which also induces a section $s_x^{\mathrm{ab}} : G_k \to \pi_1(X, \bar{z}_0)^{(\mathrm{ab})}$. Thus we have a natural map $X(k) \to H^1(G_k, TJ)$ defined by $x \mapsto [s_x^{\mathrm{ab}}]$. This map factors through $J(k)$, which is to say it coincides with the composite map

$$X(k) \xhookrightarrow{\iota} J(k) \longrightarrow H^1(G_k, TJ)$$

This is because we have an isomorphism $\pi_1(X, \bar{z}_0)^{(\mathrm{ab})} \simeq \pi_1(J, \bar{0})$ from diagram 1.5, via which the section s_x^{ab} corresponds to the section of $\pi_1(J, \bar{0})$ induced by functoriality from the rational point $\iota(x) \in J(k)$. Note that the point z_0 maps to the conjugacy class of the trivial section under this composite map.

Lemma 1.6.7. *Suppose that k has characteristic zero. Then there is an exact sequence*

$$0 \longrightarrow \widehat{J(k)} \longrightarrow H^1(G_k, TJ) \longrightarrow TH^1(G_k, J(\bar{k})) \longrightarrow 0$$

where $\widehat{J(k)} := \varprojlim_N J(k)/NJ(k)$ and $TH^1(G_k, J(\bar{k}))$ is the Tate module of the Galois cohomology group $H^1(G_k, J(\bar{k}))$.

We call this the *Kummer exact sequence.*

Proof. For each positive integer N there is a Kummer exact sequence

$$0 \longrightarrow J(\bar{k})[N] \longrightarrow J(\bar{k}) \xrightarrow{N} J(\bar{k}) \longrightarrow 0$$

This induces a long exact sequence of Galois cohomology groups

$$\cdots \longrightarrow J(k) \xrightarrow{N} J(k) \longrightarrow H^1(G_k, J(\bar{k})[N]) \longrightarrow H^1(G_k, J(\bar{k})) \xrightarrow{N} H^1(G_k, J(\bar{k})) \longrightarrow \cdots \tag{1.6}$$

hence a short exact sequence

$$0 \longrightarrow J(k)/NJ(k) \longrightarrow H^1(G_k, J(\bar{k})[N]) \longrightarrow H^1(G_k, J(\bar{k}))[N] \longrightarrow 0$$

Since the inverse system $(J(k)/NJ(k))_N$ satisfies the Mittag-Leffler condition, taking the inverse limit of this sequence over N gives the exact sequence of the Lemma. $\square$

In particular, when k has characteristic zero, the map $J(k) \to H^1(G_k, TJ)$ coincides with the inverse limit over N of the maps $J(k) \to H^1(G_k, J(\bar{k})[N])$, and sequence 1.6 in the above proof shows that each such map factors through $J(k)/NJ(k) \to H^1(G_k, J(\bar{k})[N])$. Thus the map $J(k) \to H^1(G_k, TJ)$ factors through $\widehat{J(k)}$, and we have a sequence of maps

$$X(k) \overset{\iota}{\hookrightarrow} J(k) \longrightarrow \widehat{J(k)} \hookrightarrow H^1(G_k, TJ) \tag{1.7}$$

If the map $J(k) \to \widehat{J(k)}$ is injective, different k-rational points of X give rise to distinct conjugacy classes of sections of $\pi_1(X, \bar{z}_0)^{(\mathrm{ab})}$. The same is true for sections of the étale fundamental group $\pi_1(X, \bar{z}_0)$, as the following Proposition explains.

Proposition 1.6.8. *With the above notation, suppose that k has characteristic zero, and assume that the map $J(k) \to \widehat{J(k)}$ is injective. Then two distinct k-rational points $x, x' \in X(k)$ give rise to distinct conjugacy classes of sections of $\pi_1(X, \bar{z}_0)$.*

Proof. By Proposition 1.5.6, the points $x, x' \in X(k)$ determine conjugacy classes of sections $s_x, s_{x'} : G_k \to \pi_1(X, \bar{z}_0)$. By commutativity of diagram 1.4, these induce sections $s_x^{\mathrm{ab}}, s_{x'}^{\mathrm{ab}} : G_k \to \pi_1(X, \bar{z}_0)^{(\mathrm{ab})}$ of the geometrically abelian quotient, whose conjugacy classes $[s_x^{\mathrm{ab}}]$, $[s_{x'}^{\mathrm{ab}}]$ correspond to elements of $H^1(G_k, TJ)$.

We argue by contradiction: suppose that s_x and $s_{x'}$ are conjugate. This implies that s_x^{ab} and $s_{x'}^{\mathrm{ab}}$ are conjugate, so that $[s_x^{\mathrm{ab}}] = [s_{x'}^{\mathrm{ab}}]$ in $H^1(G_k, TJ)$. The class $[s_x^{\mathrm{ab}}]$, respectively $[s_{x'}^{\mathrm{ab}}]$, is the image of x, resp. x' in the composite map

$$\begin{tikzcd}
X(k) \arrow[r, hook] & J(k) \arrow[r] & \widehat{J(k)} \arrow[r, hook] & H^1(G_k, TJ) \\
x \arrow[r, mapsto] & {[x] - [z_0]} \arrow[rr, mapsto] & & {[s_x^{\mathrm{ab}}]} \arrow[d, equal] \\
x' \arrow[r, mapsto] & {[x'] - [z_0]} \arrow[rr, mapsto] & & {[s_{x'}^{\mathrm{ab}}]}
\end{tikzcd}$$

Injectivity of the map $J(k) \to \widehat{J(k)}$ implies that the whole composite map is injective, which implies that $x = x'$. $\square$

We close this chapter with some notation concerning sections of the absolute Galois group of a curve. Let X be a smooth, geometrically connected, projective curve over a field k, and let $K := k(X)$ denote its function field. Let $\bar{\xi} : \operatorname{Spec}\Omega \to X$ be a geometric point with image the generic point ξ of X. We have a commutative diagram of exact sequences

$$\begin{tikzcd}
1 \arrow[r] & G_{X_{\bar{k}}} \arrow[r] \arrow[d, two heads] & G_X \arrow[r] \arrow[d, two heads] & G_k \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \pi_1(X_{\bar{k}}, \bar{\xi}) \arrow[r] \arrow[d, two heads] & \pi_1(X, \bar{\xi}) \arrow[r] \arrow[d, two heads] & G_k \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \pi_1(X_{\bar{k}}, \bar{\xi})^{\mathrm{ab}} \arrow[r] & \pi_1(X, \bar{\xi})^{(\mathrm{ab})} \arrow[r] & G_k \arrow[r] & 1
\end{tikzcd}$$

Let $s : G_K \to G_X$ be a section of the absolute Galois group of X. By commutativity of the above diagram, this naturally induces a section of the étale fundamental group of X, which we shall denote $s^{\text{ét}} : G_K \to \pi_1(X, \bar{\xi})$. We will call this the *étale part* of s, or the *étale section* corresponding to s.

It also induces a section of the geometrically abelian quotient, which we denote $s^{\mathrm{ab}} : G_K \to \pi_1(X, \bar{\xi})^{(\mathrm{ab})}$. We will call this the *étale abelian part* of s, or the *étale abelian section* corresponding to s.

Chapter 2

The birational section conjecture

In his 1997 paper [Tam97], Akio Tamagawa proved that the étale fundamental group of a smooth, geometrically connected, affine, hyperbolic curve over a finitely generated field over $\mathbb{Q}$ determines the curve completely up to isomorphism. He also proved a similar result for the case of smooth, affine curves over finite fields. These results were later extended by Shinichi Mochizuki in [Moc99, Theorem A] to include smooth, geometrically connected, hyperbolic curves over sub-p-adic fields. (A sub-p-adic field is any field which occurs as a sub-field of a finitely generated extension of $\mathbb{Q}_p$. This class of fields includes, in particular, the finitely generated extensions of $\mathbb{Q}$.)

Thus, when X is a smooth, geometrically connected, hyperbolic curve over a sub-p-adic field k, it should be possible to characterise the set $X(k)$ of k-rational points of X in terms of group-theoretic properties of the étale fundamental group of X. The section conjecture of Grothendieck proposes that such a characterisation is given by the sections of the fundamental exact sequence of X. In this chapter, we introduce the section conjecture and its birational variant, before stating the main results of this thesis.

2.1 The section conjecture

Let X be a smooth, projective, geometrically connected curve over a field k. Fix a geometric point $\bar{z} : \operatorname{Spec} \Omega \to X$ of X, and let $\bar{k}$ denote the algebraic closure of k in Ω. We consider the fundamental exact sequence of Theorem 1.5.1:

$$1 \longrightarrow \pi_1(X_{\bar{k}}, \bar{z}) \longrightarrow \pi_1(X, \bar{z}) \longrightarrow G_k \longrightarrow 1$$

Let us fix a universal pro-étale cover $\tilde{X} \to X$ (Definition 1.4.12). By functoriality of the fundamental group, a k-rational point $x \in X(k)$ induces a conjugacy class of sections $s_x : G_k \to \pi_1(X, \bar{z})$ of the fundamental exact sequence, and the image $s_x(G_k)$ of a section in this class is a decomposition subgroup $D_{\tilde{x}} \subset \pi_1(X, \bar{z})$ for some point $\tilde{x}$ of $\tilde{X}$ above x. (See Proposition 1.5.6, and see Definition 1.4.14 for the definition of $D_{\tilde{x}}$.) Thus we have a set-theoretic map

$$\Phi_X : X(k) \longrightarrow \overline{\mathrm{Sec}}_{\pi_1(X,\bar{z})}$$
$$x \mapsto [s_x]$$

where $\overline{\mathrm{Sec}}_{\pi_1(X,\bar{z})}$ denotes the set of conjugacy classes of sections of $\pi_1(X, \bar{z})$, with respect to conjugation by elements of $\pi_1(X_{\bar{k}}, \bar{z})$.

Definition 2.1.1. We say a section s of $\pi_1(X, \bar{z})$ is *geometric* if it is contained in a conjugacy class in the image of the map Φ_X. Equivalently, s is geometric if the image $s(G_k)$ is a decomposition group $D_{\tilde{x}}$ associated to some point $\tilde{x}$ of $\tilde{X}$ which lies above a k-rational point $x \in X(k)$.

Bijectivity of Φ_X would give us a group-theoretic characterisation of the rational points of X. One can only expect this when X is hyperbolic (Definition 1.4.8). For example, if X has genus zero and k has characteristic zero, the geometric fundamental group $\pi_1(X_{\bar{k}}, \bar{z})$ is trivial (see Example 1.4.7), hence the fundamental exact sequence becomes simply an isomorphism $\pi_1(X, \bar{z}) \simeq G_k$. Thus there exists a section of $\pi_1(X, \bar{z})$ even when X has no k-rational points.

If X is a genus 1 curve and k a number field, the geometric fundamental group is a $\hat{\mathbb{Z}}$-module (see again Example 1.4.7). Suppose that the set $X(k)$ of k-rational points of X is infinite. By the Mordell-Weil Theorem, it is a finitely generated abelian group, thus a finitely generated $\mathbb{Z}$-module. Moreover, in this case the set of conjugacy classes of sections of $\pi_1(X, \bar{z})$ is in bijection with the Galois cohomology group $H^1(G_k, \pi_1(X_{\bar{k}}, \bar{z}))$, which is a $\hat{\mathbb{Z}}$-module since $\pi_1(X_{\bar{k}}, \bar{z})$ is. If the map Φ_X is bijective then we must have a bijection $H^1(G_k, \pi_1(X_{\bar{k}}, \bar{z})) \simeq X(k)$, but this is impossible since there is no $\hat{\mathbb{Z}}$-module which is isomorphic to an infinite, finitely generated $\mathbb{Z}$-module.

So we consider only the case when X is hyperbolic. In this case, it is well-known that Φ_X is injective when k is finitely generated over $\mathbb{Q}$, and more generally when

k is a sub-p-adic field [Moc99, Theorem C]. Thus in these cases, to ask whether the section conjecture holds is to ask whether Φ_X is surjective, that is, every section of $\pi_1(X, \bar{z})$ arises from a k-rational point.

The section conjecture of Grothendieck claims that this occurs when k is finitely generated over $\mathbb{Q}$.

Conjecture 2.1.2 (Grothendieck's section conjecture, [Gro97])**.** *Let k be a finitely generated field over $\mathbb{Q}$, and let X be a smooth, geometrically connected, projective, hyperbolic curve over k. Then the map Φ_X is bijective.*

We may wish to consider the question of bijectivity of Φ_X for curves over other fields, so we make the following definition.

Definition 2.1.3. (i) Let X be a smooth, geometrically connected, projective, hyperbolic curve over a field k. We say the section conjecture *holds for X* if the map Φ_X is bijective.

(ii) For a field k, we say the section conjecture *holds over k* if the section conjecture holds for every smooth, geometrically connected, projective, hyperbolic curve over k.

The section conjecture can also be formulated for fundamental groups of affine curves. Let X be a smooth, geometrically connected, projective curve over a field k, and let $U \subseteq X$ be an open subset such that U is hyperbolic. Let $\bar{z} : \operatorname{Spec} \Omega \to U$ be a geometric point of U, and denote by $\bar{k}$ the algebraic closure of k in Ω. Fix a universal pro-étale cover $\tilde{U} \to U$, and let $\tilde{X}_U$ denote the normalisation of X in $\tilde{U}$ (Definition 1.4.13). A k-rational point of U gives rise to a section of $\pi_1(U, \bar{z})$ as above, but now the k-rational points of the complement $S := X - U$ may also induce sections of $\pi_1(U, \bar{z})$, called the *cuspidal sections* (Definition 1.5.7). The cuspidal sections are those with image in a decomposition group $D_{\tilde{x}}$ for $\tilde{x}$ a point of $\tilde{X}_U$ lying above a k-rational point x of S.

$$\begin{tikzcd}
1 \arrow[r] & I_{\tilde{x}} \arrow[r] \arrow[d, hook] & D_{\tilde{x}} \arrow[r] \arrow[d, hook] & G_{k(x)} \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \pi_1(U_{\bar{k}}, \bar{z}) \arrow[r] & \pi_1(U, \bar{z}) \arrow[r] & G_k \arrow[r] & 1
\end{tikzcd}$$

Thus the k-rational points of S may contribute sections of $\pi_1(U, \bar{z})$ which cannot be accounted for by the k-rational points of U. When k has characteristic zero, there are many conjugacy classes of sections associated to each k-rational point of S (see the discussion after Definition 1.5.7). Thus an analogue of the section conjecture for affine curves over finitely generated fields over $\mathbb{Q}$ can only ask that the k-rational points of U correspond 1-1 with the conjugacy classes of non-cuspidal sections of $\pi_1(U, \bar{z})$.

To formalise this, let $\overline{\mathrm{Sec}}_{\pi_1(U,\bar{z})}$ denote the set of conjugacy classes of sections of $\pi_1(U, \bar{z})$, and let $\overline{\mathrm{Sec}}^{\mathrm{nc}}_{\pi_1(U,\bar{z})} \subset \overline{\mathrm{Sec}}_{\pi_1(U,\bar{z})}$ denote the subset consisting of conjugacy classes of non-cuspidal sections of $\pi_1(U, \bar{z})$. We define a map

$$\Phi_U : U(k) \longrightarrow \overline{\mathrm{Sec}}^{\mathrm{nc}}_{\pi_1(U,\bar{z})}$$
$$x \mapsto [s_x]$$

where s_x arises from functoriality as above. An analogue of the section conjecture for affine curves may be stated as follows:

Let k be a finitely generated field over $\mathbb{Q}$, X a smooth, geometrically connected, projective curve over k, and $U \subseteq X$ an open subset such that U is hyperbolic. Then the map Φ_U is bijective.

2.2 The birational section conjecture

Let X be a smooth, projective, geometrically connected (not necessarily hyperbolic) curve over a field k. Fix a choice of separable closure $k(X)^{\mathrm{sep}}$ of the function field of X, and let $\bar{k}$ denote the algebraic closure of k in $k(X)^{\mathrm{sep}}$. The absolute Galois group of X

$$G_X := \mathrm{Gal}(k(X)^{\mathrm{sep}} | k(X))$$

(see Definition 1.4.20) fits into an exact sequence of Galois groups

$$1 \longrightarrow G_{X_{\bar{k}}} \longrightarrow G_X \longrightarrow G_k \longrightarrow 1$$

(see Corollary 1.5.2). Let $x \in X(k)$ be a k-rational point, and let $\tilde{x}$ be an extension of x to $k(X)^{\mathrm{sep}}$ (see Definition 1.4.22). We have a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & I_{\tilde{x}} \arrow[r] \arrow[d, hook] & D_{\tilde{x}} \arrow[r] \arrow[d, hook] & G_{k(x)} \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & G_{X_{\bar{k}}} \arrow[r] & G_X \arrow[r] & G_k \arrow[r] & 1
\end{tikzcd}$$

via which a splitting of the upper exact sequence naturally defines a section of G_X with image contained in $D_{\tilde{x}}$.

When k has characteristic zero, the upper exact sequence in the above diagram splits, hence $\tilde{x}$ gives rise to a *packet* of sections of G_X, which is isomorphic to $H^1(G_k, I_{\tilde{x}}) \simeq H^1(G_k, \hat{\mathbb{Z}}(1))$. Via the isomorphism $G_X \simeq \varprojlim_U \pi_1(U, \bar{\xi})$ from Lemma 1.4.21, where $\bar{\xi} : \operatorname{Spec} \Omega \to X$ is a geometric point with image the generic point of X such that $k(X)^{\mathrm{sep}} \subset \Omega$, such a section determines, and is determined by, a compatible system of cuspidal sections of the $\pi_1(U, \bar{\xi})$, for all open subsets $U \subset X$. (See Lemma 1.5.10 and the discussion after it.)

Definition 2.2.1. We say a section s of G_X is *geometric* if its image $s(G_k)$ is contained in a decomposition group $D_{\tilde{x}}$ for some k-rational point $x \in X(k)$ and some extension $\tilde{x}$ of x to $k(X)^{\mathrm{sep}}$. In this case, we say that the section s *arises from the point x*.

The birational analogue of the section conjecture may be stated as follows:

Let k be a finitely generated field over $\mathbb{Q}$, and let X be a smooth, projective, geometrically connected curve over k. Then every section of G_X is geometric and arises from a unique k-rational point $x \in X(k)$.

We will refer to this as the *birational section conjecture*. One may consider the statement for more general fields k, so we establish the following terminology.

Definition 2.2.2. (i) Let X be a smooth, geometrically connected, projective curve over a field k. We say the birational section conjecture *holds for X* if every section of G_X is geometric and arises from a unique k-rational point $x \in X(k)$.

(ii) For a field k, we say the birational section conjecture *holds over k* if the birational section conjecture holds for every smooth, geometrically connected, projective curve over k.

Remark 2.2.3. For a smooth, geometrically connected, projective curve X over a field k, to prove that the birational section conjecture holds for X it suffices to prove that every section of G_X arises from a k-rational point $x \in X(k)$. The uniqueness of this point x is automatic, since decomposition subgroups of G_X associated to distinct valuations of $k(X)^{\mathrm{sep}}$ intersect trivially. This statement may be found in [NSW08, Corollary 12.1.3], though here only the case of global fields is discussed, when in fact the proof works for any field.

In [Koe05], Koenigsmann proved that the birational section conjecture holds over finite extensions of $\mathbb{Q}_p$ and over $\mathbb{R}$. The section conjecture is still open for all base fields, though there is hope that the birational section conjecture might be used to deduce the section conjecture in some cases, via the theory of "cuspidalisation". Conversely, the section conjecture implies the birational section conjecture, as we now explain.

Definition 2.2.4. Let X be a smooth, projective, geometrically connected curve over a field k. A *neighbourhood* of a section $s : G_k \to G_X$ is an open subgroup $H \subset G_X$ which contains $s(G_k)$.

For an open subgroup $H \subset G_X$, let us write $X_H \to X$ for the finite morphism corresponding to H, so that $G_{X_H} = H$. The following observation is due to Tamagawa (see [Tam97, Proposition 2.8 (iv)].)

Lemma 2.2.5 (Limit Argument). *Let k be a field which admits a structure of a Hausdorff topological field such that $Y(k)$ is compact for any smooth, geometrically connected, projective, hyperbolic curve Y over k. Let X be a smooth, geometrically connected, projective curve over k, and let $s : G_k \to G_X$ be a section. Then s is geometric if and only if $X_H(k) \neq \emptyset$ for every neighbourhood H of s.*

Proof. First assume s is geometric. Let $x \in X(k)$ be the k-rational point and $\tilde{x}$ the extension of x to $k(X)^{\mathrm{sep}}$ such that $s(G_k)$ is contained in the decomposition subgroup $D_{\tilde{x}} \subset G_X$. For a neighbourhood H of s, let $x_H \in X_H$ denote the point corresponding to the restriction of $\tilde{x}$ to $k(X_H)$. Then $\mathrm{Gal}(\bar{k}|k(x_H))$ coincides with the image of $D_{\tilde{x}} \cap H$ under the projection $G_X \twoheadrightarrow G_k$. But since H is a neighbourhood of s, $D_{\tilde{x}} \cap H$ must map surjectively to G_k, hence $k(x_H) = k$ and $X_H(k) \neq \emptyset$.

Next assume $X_H(k) \neq \emptyset$ for every neighbourhood H of s. We have an identity

$$X_{s(G_k)}(k) = \varprojlim_{s(G_k) \subset H \subset G_X} X_H(k) \tag{2.1}$$

where the limit is taken over the neighbourhoods of s, ordered by inclusion.

Claim. There exists a neighbourhood H of s such that X_H is hyperbolic, and then $X_{H'}$ is hyperbolic for every neighbourhood H' contained in H.

Proof of Claim. First we wish to find a finite, separable morphism $f : \overline{Y} \to X_{\bar{k}}$ such that $\overline{Y}$ is a smooth, connected, projective *hyperbolic* curve over $\bar{k}$ and f is *tamely ramified*, which is to say the ramification indices are all prime to the characteristic of k. Let $g(X_{\bar{k}})$, respectively $g(\overline{Y})$ denote the genus of $X_{\bar{k}}$, resp. of $\overline{Y}$. Recall from Definition 1.4.8 that a smooth, geometrically connected, projective curve is hyperbolic if its genus is at least 2.

The Hurwitz formula [Liu02, Theorem 7.4.16] implies that $g(\overline{Y}) \geq g(X_{\bar{k}})$ (see also [Liu02, Corollary 7.4.19]). Thus if $g(X_{\bar{k}}) \geq 2$ then we may take f to be any finite, separable, tamely ramified morphism of smooth, connected, projective curves.

If $g(X_{\bar{k}}) = 1$ and f is separable and tamely ramified, the Hurwitz formula implies that $g(\overline{Y}) = 1 + \frac{1}{2}\sum_{y \in \overline{Y}}(e_y - 1)$, where e_y is the ramification index of f at y. Thus $g(\overline{Y})$ is independent of the degree of f, and $g(\overline{Y}) \geq 2$ if f is ramified in at least 2 points. Thus we may take f to be any finite, separable morphism of any degree such that $\overline{Y}$ is a smooth, connected, projective curve and f is tamely ramified in at least 2 points.

If $g(X_{\bar{k}}) = 0$ then $X_{\bar{k}}$ is isomorphic to $\mathbb{P}^1_{\bar{k}}$. Indeed, the assumption that $X_H(k) \neq \emptyset$ for every neighbourhood H of s implies in particular that $X(k)$ and therefore $X_{\bar{k}}(\bar{k})$ is non-empty, thus $X_{\bar{k}} \simeq \mathbb{P}^1_{\bar{k}}$ by [Liu02, Proposition 7.4.1]. Then we may choose f to be the finite morphism corresponding to the polynomial $s^m = (t - \alpha_1) \cdots (t - \alpha_r)$, where $m \geq 2$ is prime to the characteristic of k, $r \geq 2(m+1)/(m-1)$ and $\alpha_1, \ldots, \alpha_r$ are any r distinct elements of $\bar{k}$. Then f is a finite, separable morphism of degree m which is tamely ramified in r points with ramification indices all equal to m, thus $g(\overline{Y}) \geq 2$ by the Hurwitz formula.

With the morphism $f : \overline{Y} \to X_{\bar{k}}$ suitably chosen as above, let $U_{\bar{k}} \subset X_{\bar{k}}$ denote the open subset over which f is unramified, and let $\bar{\xi} : \operatorname{Spec} \Omega \to X_{\bar{k}}$ denote a geometric point with image the generic point of $X_{\bar{k}}$. Then the finite morphism

$\overline{Y} \to X_{\bar{k}}$ corresponds to an open subgroup $\overline{H} \subseteq \pi_1^t(U_{\bar{k}}, \bar{\xi})$, where the superscript 't' denotes that this is the *tame fundamental group* of $U_{\bar{k}}$. We will not define this object explicitly here - suffice to say it characterises finite morphisms to $X_{\bar{k}}$ which are étale over $U_{\bar{k}}$ and at worst tamely ramified over the points of $X_{\bar{k}} - U_{\bar{k}}$. In particular, if k has characteristic zero then this is just the usual étale fundamental group of $U_{\bar{k}}$. See, for example, [Sza09, Definition 5.7.15] for a formal definition.

By [Tam97, Proposition 1.1] the group $\pi_1^t(U_{\bar{k}}, \bar{\xi})$ is finitely generated, hence by [RZ10, Proposition 2.5.1] there is a fundamental system of neighbourhoods of the identity element in $\pi_1^t(U_{\bar{k}}, \bar{\xi})$ consisting of a countable chain of open characteristic subgroups

$$\pi_1^t(U_{\bar{k}}, \bar{\xi}) = H_0 \supseteq H_1 \supseteq H_2 \supseteq \cdots$$

Thus there exists n such that the n-th characteristic subgroup H_n in this chain satisfies $H_n \subseteq \overline{H}$, so that H_n corresponds to a finite morphism $\overline{Y}_n \to \overline{Y}$ of smooth, connected, projective curves, and therefore to a finite morphism $\bar{f}_n : \overline{Y}_n \to X_{\bar{k}}$. Moreover, $\overline{Y}_n$ also has genus at least 2 (see above argument and [Liu02, Corollary 7.4.19]).

Let $s_U : G_k \to \pi_1^t(U, \bar{\xi})$ denote the section of $\pi_1^t(U, \bar{\xi})$ induced by s via the natural quotient homomorphism $G_X \twoheadrightarrow \pi_1^t(U, \bar{\xi})$. Then the semidirect product $H_n \rtimes s_U(G_k) \subset \pi_1^t(U, \bar{\xi})$ defines a neighbourhood of the section s_U. Moreover, denoting by $f_n : Y_n \to X$ the finite morphism corresponding to $H_n \rtimes s_U(G_k)$, Y_n also has genus at least 2 by the Hurwitz formula - indeed, we have an equality of indices $[H_n \rtimes s_U(G_k) : \pi_1^t(U, \bar{\xi})] = [H_n : \pi_1^t(U_{\bar{k}}, \bar{\xi})]$ and thus an equality of degrees $\deg f_n = \deg \bar{f}_n$, and the ramification indices of f_n and $\bar{f}_n$ are the same since only the base field is extended.

The preimage of $H_n \rtimes s_U(G_k)$ in G_X is then a neighbourhood of the section s, corresponding to the same finite morphism $f_n : Y_n \to X$. For any neighbourhood $H' \subseteq H_n \rtimes s_U(G_k)$, $X_{H'} \to Y_n$ is a finite morphism, thus $X_{H'}$ is also hyperbolic by the Hurwitz formula. □

Let H be a neighbourhood of s such that X_H is hyperbolic. Then, by the above Claim and our assumption on k, for every neighbourhood H' contained in H, $X_{H'}(k)$ is compact. Thus it follows from identity 2.1 that $X_{s(G_k)}(k) \neq \emptyset$.

Let $x \in X_{s(G_k)}(k)$, and choose an extension $\tilde{x}$ of x to $k(X)^{\mathrm{sep}}$. Then $D_{\tilde{x}} \cap s(G_k)$ maps surjectively to G_k under the projection $G_X \twoheadrightarrow G_k$, since x is k-rational. Since

$s(G_k) \to G_k$ is injective, this implies $D_{\tilde{x}} \cap s(G_k) = s(G_k)$, i.e. $s(G_k) \subset D_{\tilde{x}}$ and s is geometric. $\square$

Proposition 2.2.6. *Let k be a field which admits the structure of a Hausdorff topological space such that for any smooth, geometrically connected, projective, hyperbolic curve Y over k, $Y(k)$ is compact. If the section conjecture holds over k, then the birational section conjecture holds over k.*

Proof. Assume the section conjecture holds over k, and let X be any smooth, geometrically connected, projective (not necessarily hyperbolic) curve over k. Let $s : G_k \to G_X$ be a section of the absolute Galois group of X. Let $H \subset G_X$ be a neighbourhood of s, $X_H \to X$ the corresponding finite morphism and $\bar{\eta} : \operatorname{Spec} \Omega_H \to X_H$ a geometric point with image the generic point of X_H. Then s induces a section $s_H : G_k \to H = G_{X_H}$, which in turn induces a section $s_H^{\text{ét}} : G_k \to \pi_1(X_H, \bar{\eta})$ of the étale fundamental group of X_H.

$$
\begin{tikzcd}
G_{X_H} \arrow[r] \arrow[d, two heads] & G_X \arrow[r] \arrow[d, two heads] & G_k \arrow[l, bend right, "s"'] \arrow[ll, bend right, "s_H"'] \arrow[d, equal] \\
\pi_1(X_H, \bar{\eta}) \arrow[r] & \pi_1(X, \bar{\xi}) \arrow[r] & G_k \arrow[ll, bend right, "s_H^{\text{ét}}"']
\end{tikzcd}
$$

If X_H is hyperbolic then, by the assumption that the section conjecture holds over k, the section $s_H^{\text{ét}}$ is geometric, which means in particular that there exists a k-rational point $x \in X_H(k)$. If X_H is not hyperbolic, there exists another neighbourhood $H' \subset H \subset G_X$ of s such that $X_{H'}$ is hyperbolic (see the Claim in the proof of Lemma 2.2.5). The above argument implies there exists $x' \in X_{H'}(k)$, and the image of x' in X_H is necessarily also k-rational. Thus $X_H(k) \neq \emptyset$ for any neighbourhood H of s, which by Lemma 2.2.5 implies that s is geometric. This proves that the birational section conjecture holds for X (see Remark 2.2.3), thus, since X was arbitrary, the birational section conjecture holds over k. $\square$

2.3 Statement of the Main Theorems

In this study we investigate the birational section conjecture over function fields. We prove that, for a certain class of fields k of characteristic zero, and under the condition of finiteness of certain Shafarevich-Tate groups, proving that the birational section conjecture holds over finitely generated extensions of k reduces to proving that it holds over finite extensions of k. This class of fields contains, in particular, the finitely generated extensions of $\mathbb{Q}$, so the result reduces the birational section conjecture over such fields to the case of number fields, under the finiteness condition on the Shafarevich-Tate groups. Our approach is an adaptation of the proof by Mohamed Saïdi in [Saï16] of a similar result for the section conjecture.

Let us start by describing the aforementioned class of fields, which was introduced in [Saï16, Definition 0.2]. See Definition 1.6.5 for the definitions of the Tate module and ℓ-adic Tate module.

Definition 2.3.1. For a field k' of characteristic zero, consider the following conditions on k'.

(i) The birational section conjecture holds over k'.

(ii) For every prime integer ℓ', the ℓ'-cyclotomic character $\chi_{\ell'} : G_{k'} \to \mathbb{Z}_{\ell'}^{\times}$ is non-Tate, meaning that any $G_{k'}$-map $\mathbb{Z}_{\ell'}(1) \to T_{\ell'}A$, for some abelian variety A, vanishes.

(iii) Given an abelian variety A over k', any quotient $A(k') \twoheadrightarrow D$ of the group of k'-rational points $A(k')$ satisfies the following:

   (a) The natural map $D \to \widehat{D}$ is injective, where $\widehat{D} := \varprojlim_N D/ND$ (see Notation).

   (b) The torsion group $D[N]$ (see Notation) is finite for all $N \geq 1$, and the Tate module TD is trivial.

(iv) Given a separated, smooth, connected curve C over k' with function field $K = k'(C)$, K admits the structure of a Hausdorff topological field, so that $X(K)$ is compact for any smooth, geometrically connected, projective, hyperbolic curve X over K.

(v) Given a smooth and connected (not necessarily projective) curve C over k' with function field $K = k'(C)$ and a finite morphism $\tilde{C} \to C$, then the following holds. If $\tilde{C}_c(k'(c)) \neq \emptyset$ for all $c \in C^{\mathrm{cl}}$, where $k'(c)$ denotes the residue field at

c and $\tilde{C}_c$ is the inverse image of c in $\tilde{C}$, then $\tilde{C}(K) \neq \emptyset$.

For a field k of characteristic zero, we say that k *strongly satisfies* one of the above conditions (i), (ii), (iii), (iv) and (v) if this condition is satisfied by any finite extension $k'|k$ of k.

Condition (i) is the strongest of the above conditions - conditions (ii)-(v) are much milder conditions which in particular are satisfied by finitely generated fields over $\mathbb{Q}$. Condition (v) is satisfied by Hilbertian fields (see [Saï16, Lemma 4.1.5]).

Definition 2.3.2. Let k be a field of characteristic zero and C a smooth, separated, connected curve over k with function field K. Let $\mathcal{A} \to C$ be an abelian scheme with generic fibre $A = \mathcal{A} \times_C \operatorname{Spec} K$. For each closed point $c \in C^{\mathrm{cl}}$ denote by K_c the completion of K at c, and write $A_c = A \times_{\operatorname{Spec} K} \operatorname{Spec} K_c$. Let $\overline{K_c}$ be an algebraic closure of K_c and $\overline{K}$ the algebraic closure of K inside $\overline{K_c}$. We define the Shafarevich-Tate group

$$Ш(\mathcal{A}) = \ker(H^1(G_K, A(\overline{K})) \to \prod_{c \in C^{\mathrm{cl}}} H^1(G_{K_c}, A_c(\overline{K_c})))$$

where the product is taken over all the closed points of C.

We now state our two main theorems. See Definition 1.1.3 for the definition of a relative curve (which is, in particular, flat and proper) and Remark 1.1.10 for the meaning of the relative Jacobian.

Theorem A. *Let k be a field of characteristic zero that strongly satisfies conditions (i)-(v) of Definition 2.3.1. Let C be a smooth, separated, connected curve over k with function field K. Let $\mathcal{X} \to C$ be a smooth relative curve, with generic fibre $X := \mathcal{X} \times_C \operatorname{Spec} K$ which is a geometrically connected hyperbolic curve over K such that $X(K) \neq \emptyset$. Denoting by $\mathcal{J} := \operatorname{Pic}^0_{\mathcal{X}/C}$ the relative Jacobian of $\mathcal{X}$, assume $TШ(\mathcal{J}) = 0$. Then the birational section conjecture holds for X.*

Theorem B. *Let k, C and K be as in Theorem A. For any finite extension L of K, let C^L denote the normalisation of C in L. Assume that for any such finite extension L and any smooth relative curve $\mathcal{Y} \to C^L$ we have $TШ(\mathcal{J}_{\mathcal{Y}}) = 0$, where $\mathcal{J}_{\mathcal{Y}} := \operatorname{Pic}^0_{\mathcal{Y}/C^L}$ is the relative Jacobian of $\mathcal{Y}$. Then the birational section conjecture holds over all finite extensions of K.*

In the context of $\mathbb{Q}$, this means that if the birational section conjecture holds over all fields of some fixed transcendence degree over $\mathbb{Q}$ then it holds over all fields of transcendence degree one higher, provided that $Ш(\mathcal{J}_\mathcal{Y})$ is finite for all such fields L and all relative Jacobians $\mathcal{J}_\mathcal{Y}$ as described in the statement of Theorem B. By induction this means that, under the assumption on the relevant Shafarevich-Tate groups, if the birational section conjecture holds over number fields then it holds over all finitely generated fields over $\mathbb{Q}$.

Chapter 3

Specialisation of Sections

In this chapter we will consider curves over local fields. Let R be a complete discrete valuation ring with uniformiser π, field of fractions K and residue field $k := R/\pi R$. Assume further that k has characteristic zero. Let X be a smooth, geometrically connected *relative curve* over R, and let X_K, resp. X_k denote its generic fibre, resp. its closed fibre. Given a section $s : G_K \to G_{X_K}$ of the absolute Galois group of X_K (Definition 1.4.20), our aim is to show that, under the assumption that the birational section conjecture holds over k (Definition 2.2.2), s gives rise naturally to a k-rational point $x \in X(k)$ of the closed fibre X_k.

To achieve this we will investigate when such a section s naturally induces a section $\bar{s} : G_k \to G_{X_k}$ of the absolute Galois group of the closed fibre X_k, which we call the *specialisation* of s. To this end we define a specialisation homomorphism of absolute Galois groups by following the definition of the specialisation homomorphism for fundamental groups of affine curves (an exposition of which can be found in [OV98]), then applying the relation in Lemma 1.4.21.

The fundamental group of an open subset $U_k \subset X_k$ contains inertia groups corresponding to the points of the complement $S_k := X_k - U_k$. These inertia groups play an important role in the specialisation of sections, giving rise to the phenomenon of *ramification* of sections. We impose conditions on k which imply that to each section of G_{X_K} we can naturally associate a k-rational point of X_k, whether or not it is ramified.

3.1 The specialisation homomorphism: projective case

We will first recall the definition of the specialisation homomorphism for fundamental groups of projective curves using formal geometry. Further details on this theory can be found in [GR71, Exposé IX] and [OV98, Chapitre 11].

Let R be a complete discrete valuation ring with uniformiser π, field of fractions K and residue field $k := R/\pi R$. Let X be a smooth, geometrically connected *relative curve* over $\operatorname{Spec} R$ (see Definition 1.1.3), and denote by X_K its generic fibre and X_k its closed fibre. Fix an algebraic closure $\overline{K}$ of K. Denote by $\overline{R}$ the integral closure of R in $\overline{K}$, and by $\bar{k}$ the residue field of $\overline{R}$, which is an algebraic closure of k.

Let $X_{\overline{K}} := X \times_{\operatorname{Spec} R} \operatorname{Spec} \overline{K}$ denote the geometric fibre of X_K, and similarly write $X_{\overline{R}} := X \times_{\operatorname{Spec} R} \operatorname{Spec} \overline{R}$ (see Notation). We have a diagram:

$$\begin{array}{ccccc} X_{\overline{K}} & \longrightarrow & X_K & \longrightarrow & \operatorname{Spec} K \\ \downarrow & & \downarrow & & \downarrow \\ X_{\overline{R}} & \longrightarrow & X & \longrightarrow & \operatorname{Spec} R \end{array}$$

Let $\bar{z}_1 : \operatorname{Spec} \Omega_1 \to X_{\overline{K}}$ be a geometric point of $X_{\overline{K}}$. By composition this defines geometric points of X_K, $X_{\overline{R}}$ and X, which we again denote by $\bar{z}_1$. Similarly, it also defines a geometric point $\bar{\eta} : \operatorname{Spec} \Omega_1 \to \operatorname{Spec} R$, whose image is the generic point η of $\operatorname{Spec} R$, corresponding to the zero ideal of R.

Functoriality of the fundamental group yields a diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(X_{\overline{K}}, \bar{z}_1) & \longrightarrow & \pi_1(X_K, \bar{z}_1) & \longrightarrow & G_K & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \pi_1(X_{\overline{R}}, \bar{z}_1) & \longrightarrow & \pi_1(X, \bar{z}_1) & \longrightarrow & \pi_1(\operatorname{Spec} R, \bar{\eta}) & \longrightarrow & 1 \end{array} \tag{3.1}$$

where the upper row is the fundamental exact sequence of X_K (see Theorem 1.5.1).

Remark 3.1.1. (i) Exactness of the lower row in the above diagram follows from the same argument used to show exactness of the fundamental exact sequence of a quasi-compact, geometrically integral scheme over a field. See, for example, [Sza09, Proposition 5.6.1] for an outline of this argument.

(ii) The homomorphism $G_K \to \pi_1(\operatorname{Spec} R, \bar{\eta})$ is surjective. Indeed, the étale Galois covers of $\operatorname{Spec} R$ are exactly the morphisms of the form $\operatorname{Spec} R' \to \operatorname{Spec} R$

where $K' := \mathrm{Frac}(R')$ is a Galois extension of K which is unramified with respect to π. By Lemma 1.4.5, we therefore have

$$\pi_1(\operatorname{Spec} R, \bar{\eta}) \simeq \varprojlim_{\substack{\mathrm{Frac}(R')|K \\ \text{unramified}}} \mathrm{Aut}(R'|R) \simeq \varprojlim_{\substack{K'|K \\ \text{unramified}}} \mathrm{Gal}(K'|K) \simeq \mathrm{Gal}(K^{\mathrm{ur}}|K)$$

where K^{ur} denotes the maximal unramified extension of K. Hence $\pi_1(\operatorname{Spec} R, \bar{\eta})$ is a quotient of G_K.

Lemma 3.1.2. (i) *The homomorphisms* $\pi_1(X_{\overline{K}}, \bar{z}_1) \to \pi_1(X_{\overline{R}}, \bar{z}_1)$ *and* $\pi_1(X_K, \bar{z}_1) \to \pi_1(X, \bar{z}_1)$ *are surjective.*

(ii) *The homomorphism* $\pi_1(X_{\overline{K}}, \bar{z}_1) \to \pi_1(X_{\overline{R}}, \bar{z}_1)$ *is an isomorphism when k has characteristic zero.*

Proof. To prove that $\pi_1(X_K, \bar{z}_1) \to \pi_1(X, \bar{z}_1)$ is surjective it suffices, by Proposition 1.4.11 (i), to show that for any connected étale cover $Y \to X$ the fibre product $Y \times_X X_K = Y \times_{\operatorname{Spec} R} \operatorname{Spec} K = Y_K$ is also connected. If Y is connected then, since it is regular, it is irreducible. Since Y_K is an open subset of Y (its complement is the closed fibre Y_k), it is also irreducible, hence connected. The same argument shows that $\pi_1(X_{\overline{K}}, \bar{z}_1) \to \pi_1(X_{\overline{R}}, \bar{z}_1)$ is surjective.

We give an outline of the proof of statement (ii): for more details see [GR71, Exposé X, Corollaire 3.9]. To prove the statement it suffices, by Proposition 1.4.11 (ii), to show that every étale cover $Y_{\overline{K}} \to X_{\overline{K}}$ comes from an étale cover of $X_{\overline{R}}$, that is, there exists an étale cover $Y_{\overline{R}} \to X_{\overline{R}}$ such that $Y_{\overline{K}} = Y_{\overline{R}} \times_{\operatorname{Spec} \overline{R}} \operatorname{Spec} \overline{K}$. By [Sza09, Lemma 5.6.2], $Y_{\overline{K}} \to X_{\overline{K}}$ descends to an étale cover of X_L for some finite extension $L|K$ in $\overline{K}$, that is, there exists an étale cover $Y_L \to X_L$ such that $Y_{\overline{K}} = Y_L \times_{\operatorname{Spec} L} \operatorname{Spec} \overline{K}$. It then suffices to show that there exists a finite extension $L'|L$ in $\overline{K}$ such that the étale cover $Y_{L'} \to X_{L'}$ induced from $Y_L \to X_L$ by base change to L' comes from an étale cover $Y_{R'} \to X_{R'}$, where R' denotes the integral closure of R in L'. Indeed, then we may take $Y_{\overline{R}} := Y_{R'} \times_{\operatorname{Spec} R'} \operatorname{Spec} \overline{R}$.

For any finite extension $L'|L$ we may take the normalisation $Y_{R'}$ of $X_{R'}$ in the function field $k(Y_{L'})$, but we need to choose L' so that the finite morphism $Y_{R'} \to X_{R'}$ is unramified over the generic point ξ' of $X_{l'}$, where l' denotes the residue field of R'. Then the branch locus of $Y_{R'} \to X_{R'}$ is either empty or of codimension 2 (consisting of finitely many closed points of $X_{l'}$). By Zariski's Purity Theorem (Theorem 1.3.5), it must therefore be empty, hence $Y_{R'} \to X_{R'}$ is unramified everywhere.

Since ξ' has codimension 1 in $X_{R'}$, the local ring $\mathcal{O}_{X_{R'},\xi'}$ is a discrete valuation ring, and its field of fractions is the function field $k(X_{L'})$. Abhyankar's Lemma [GR71, Exposé X, Lemme 3.6] then implies that we can choose L' so that, for each $\zeta' \in Y_{R'}$ mapping to ξ', the local ring $\mathcal{O}_{Y_{R'},\zeta'}$, which has field of fractions $k(Y_{L'})$, is unramified over $\mathcal{O}_{X_{R'},\xi'}$. That is, $Y_{R'} \to X_{R'}$ is unramified over ξ', which completes the proof. □

Now let $\bar{z}_2 : \operatorname{Spec} \Omega_2 \to X_{\bar{k}}$ be a geometric point of $X_{\bar{k}}$. By composition this defines geometric points of X_k, $X_{\overline{R}}$ and X, which we again denote by $\bar{z}_2$. Similarly, it also defines a geometric point $\bar{\pi} : \operatorname{Spec} \Omega_2 \to \operatorname{Spec} R$, whose image is the unique closed point of $\operatorname{Spec} R$ corresponding to the maximal ideal $\pi R \subset R$. Functoriality again yields a diagram:

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(X_{\overline{R}}, \bar{z}_2) \arrow[r] & \pi_1(X, \bar{z}_2) \arrow[r] & \pi_1(\operatorname{Spec} R, \bar{\pi}) \arrow[r] & 1 \\
1 \arrow[r] & \pi_1(X_{\bar{k}}, \bar{z}_2) \arrow[r] \arrow[u] & \pi_1(X_k, \bar{z}_2) \arrow[r] \arrow[u] & G_k \arrow[r] \arrow[u] & 1
\end{tikzcd} \tag{3.2}$$

where the lower row is the fundamental exact sequence of X_k.

Remark 3.1.3. Note that the homomorphism $G_k \to \pi_1(\operatorname{Spec} R, \bar{\pi})$ is an isomorphism. Indeed, we have an isomorphism $\pi_1(\operatorname{Spec} R, \bar{\pi}) \simeq \operatorname{Gal}(K^{\mathrm{ur}}|K)$ by Remark 3.1.1 and Theorem 1.4.10, and we have $\operatorname{Gal}(K^{\mathrm{ur}}|K) \simeq G_k$ since R is a complete discrete valuation ring.

Theorem 3.1.4. *The homomorphism $\pi_1(X_{\bar{k}}, \bar{z}_2) \to \pi_1(X_{\overline{R}}, \bar{z}_2)$ is an isomorphism.*

Proof. We give here an outline of the proof - for more details see [OV98]. First we prove surjectivity. By Proposition 1.4.11 (i) it suffices to show that, for any étale cover $Y \to X_{\overline{R}}$ with Y connected, $Y_{\bar{k}}$ is also connected. For this we employ the Stein factorisation $Y \to Y' \to \operatorname{Spec} \overline{R}$, in which Y' is finite over $\operatorname{Spec} \overline{R}$ and the morphism $Y \to Y'$ has non-empty, geometrically connected fibres (see [Sza09, Facts 5.6.5]). In particular, there is a bijection between the connected components of $Y_{\bar{k}}$ and those of $Y'_{\bar{k}}$. Since $Y' \to \operatorname{Spec} \overline{R}$ is finite and $\overline{R}$ is Henselian, Y' must be of the form $\prod_{i=1}^{n} \operatorname{Spec} R_i$, where each R_i is a local ring finite over $\overline{R}$. But since Y is connected, Y' is also connected, hence $n = 1$. This implies that $Y'_{\bar{k}}$ is connected, hence so is $Y_{\bar{k}}$.

For injectivity, by Proposition 1.4.11 (ii) it suffices to show that any given étale cover $\overline{Y} \to X_{\bar{k}}$ lifts to an étale cover $Y \to X_{\overline{R}}$ such that $Y_{\bar{k}} = \overline{Y}$. There exists a finite extension $k'|k$ such that $\overline{Y} \to X_{\bar{k}}$ descends to an étale cover $Y_{k'} \to X_{k'}$ [Sza09, Lemma 5.6.2]. Thus, denoting by R' a finite extension of R with residue field k', it suffices to show that $Y_{k'} \to X_{k'}$ lifts to an étale cover $Y_{R'} \to X_{R'}$, since then $Y := Y_{R'} \times_{\operatorname{Spec} R'} \operatorname{Spec} \overline{R} \to X_{\overline{R}}$ is the desired cover of $X_{\overline{R}}$ which lifts $\overline{Y} \to X_{\bar{k}}$.

One proves this by passing to formal geometry. Let $\hat{X}_{R'}$ denote the formal completion of $X_{R'}$ along the special fibre $X_{k'}$, which is obtained by taking the projective limit $\varprojlim_n X_n$, where $X_n = X_{R'} \times_{\operatorname{Spec} R'} \operatorname{Spec} R'/(\pi')^{n+1}$, π' being a uniformiser of R'. The formal scheme $\hat{X}_{R'}$ can be interpreted as a "formal neighbourhood" of $X_{k'}$ in $X_{R'}$. By [Gro67, Ch. IV, Théorème 18.1.2], the functor $Y_n \mapsto Y_n \times_{X_n} X_{k'}$ defines an equivalence of categories between the étale covers of X_n and those of $X_{k'}$. We thus obtain a formal étale cover $\hat{Y}_{R'} \to \hat{X}_{R'}$ of the formal scheme $\hat{X}_{R'}$ uniquely determined by $Y_{k'} \to X_{k'}$. Grothendieck's existence theorem [Gro67, Ch. III, Théorème 5.1.4] tells us that $\hat{Y}_{R'}$ can be uniquely "algebraised" to an étale cover $Y_{R'} \to X_{R'}$, which satisfies $Y_{R'} \times_{\operatorname{Spec} R'} \operatorname{Spec} k' = Y_{k'}$ by construction. $\square$

Corollary 3.1.5. *The homomorphism $\pi_1(X_k, \bar{z}_2) \to \pi_1(X, \bar{z}_2)$ is an isomorphism.*

Proof. This follows easily from Theorem 3.1.4 and Remark 3.1.3 via an elementary diagram chase. $\square$

By Theorem 1.4.10, there exist isomorphisms $\pi_1(X, \bar{z}_1) \simeq \pi_1(X, \bar{z}_2)$, $\pi_1(X_{\overline{R}}, \bar{z}_1) \simeq \pi_1(X_{\overline{R}}, \bar{z}_2)$ and $\pi_1(\operatorname{Spec} R, \bar{\eta}) \simeq \pi_1(\operatorname{Spec} R, \bar{\pi})$, unique up to inner automorphism. Thus we can combine diagrams 3.1 and 3.2, to give a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(X_{\overline{K}}, \bar{z}_1) \arrow[r] \arrow[d, two heads] & \pi_1(X_K, \bar{z}_1) \arrow[r] \arrow[d, two heads] & G_K \arrow[r] \arrow[d, two heads] & 1 \\
1 \arrow[r] & \pi_1(X_{\overline{R}}, \bar{z}_1) \arrow[r] \arrow[d, phantom, "\wr"] & \pi_1(X, \bar{z}_1) \arrow[r] \arrow[d, phantom, "\wr"] & \pi_1(\operatorname{Spec} R, \bar{\eta}) \arrow[r] \arrow[d, phantom, "\wr"] & 1 \\
1 \arrow[r] & \pi_1(X_{\overline{R}}, \bar{z}_2) \arrow[r] & \pi_1(X, \bar{z}_2) \arrow[r] & \pi_1(\operatorname{Spec} R, \bar{\pi}) \arrow[r] & 1 \\
1 \arrow[r] & \pi_1(X_{\bar{k}}, \bar{z}_2) \arrow[r] \arrow[u, "\wr"] & \pi_1(X_k, \bar{z}_2) \arrow[r] \arrow[u, "\wr"] & G_k \arrow[r] \arrow[u, "\wr"] & 1
\end{tikzcd} \tag{3.3}$$

where the uppermost and lowermost rows are the fundamental exact sequences. By composing the vertical homomorphisms we obtain surjective homomorphisms $\overline{\mathrm{Sp}}_X$, Sp_X and ρ making the following diagram commutative:

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(X_{\overline{K}}, \bar{z}_1) \arrow[r] \arrow[d, two heads, "\overline{\mathrm{Sp}}"'] & \pi_1(X_K, \bar{z}_1) \arrow[r] \arrow[d, two heads, "\mathrm{Sp}"'] & G_K \arrow[r] \arrow[d, two heads, "\rho"'] & 1 \\
1 \arrow[r] & \pi_1(X_{\bar{k}}, \bar{z}_2) \arrow[r] & \pi_1(X_k, \bar{z}_2) \arrow[r] & G_k \arrow[r] & 1
\end{tikzcd}$$

The homomorphisms $\overline{\mathrm{Sp}}_X$ and Sp_X are called *specialisation homomorphisms of fundamental groups.* They are defined only up to inner automorphism of $\pi_1(X_{\overline{R}}, \bar{z}_2)$, resp. $\pi_1(X, \bar{z}_2)$, due to the choices of base points $\bar{z}_1$, $\bar{z}_2$. By Lemma 3.1.2 (ii), the homomorphism $\overline{\mathrm{Sp}}_X$ is an isomorphism when k has characteristic zero.

Note that, since R is a complete discrete valuation ring, G_K is the decomposition group associated to the unique prime ideal of R, hence the surjection ρ is the natural quotient map - see, for example, [Ser79, Ch. 1, §7, Proposition 20].

3.2 The specialisation homomorphism: affine case

As in the previous section, let R be a complete discrete valuation ring with uniformiser π, field of fractions K and residue field k, and let X be a smooth, geometrically connected relative curve over $\operatorname{Spec} R$. We will consider complements of divisors D on X which are finite étale over R, that is, such that the composite morphism $D \hookrightarrow X \to \operatorname{Spec} R$ is a finite, étale morphism.

Lemma 3.2.1. *A divisor D on X is finite étale over R if and only if the following conditions hold:*

(i) *D_K and D_k consist of the same finite number of closed points;*

(ii) *The points of D_K have residue fields which are unramified extensions of K;*

In particular, if D is finite étale over R then for each closed point $x \in D_k$ there is exactly one closed point of D_K which specialises to x.

Proof. D is finite étale over R if and only if $D \simeq \operatorname{Spec}(\oplus_{i=1}^n R_i) \simeq \cup_{i=1}^n \operatorname{Spec} R_i$ for some n, where each R_i is a ring whose field of fractions K_i is an unramified extension of K. Then $D_K \simeq \cup_{i=1}^n \operatorname{Spec} K_i$, which is to say D_K consists of closed points $x_1, \ldots, x_n$ having residue fields $K_1, \ldots, K_n$ respectively. Since $\operatorname{Spec} R_i \to \operatorname{Spec} R$ is unramified for every i, we have $D_k \simeq \cup_{i=1}^n \operatorname{Spec} k_i$, where k_i denotes the residue field of R_i, hence D_k also consists of n closed points of X_k. Note that the point of D_K corresponding to $\operatorname{Spec} K_i$ specialises to the point of D_k corresponding to $\operatorname{Spec} k_i$. $\square$

Let S be a divisor on X which is finite étale over R, and let $U := X - S$ denote the complement of S in X, which is an open subset of X. Then U_K is an open subset of the generic fibre X_K of X with complement S_K, and likewise U_k is an open subset of the closed fibre X_k with complement S_k.

Fix an algebraic closure $\overline{K}$ of K. Denote by $\overline{R}$ the integral closure of R in $\overline{K}$, and by $\bar{k}$ the residue field of $\overline{R}$, which is an algebraic closure of k. Note that $S_{\overline{R}}$ is finite étale over $\operatorname{Spec} \overline{R}$ (since finite and étale morphisms are stable under base change), and it is the complement of $U_{\overline{R}}$ in $X_{\overline{R}}$.

Let $\bar{z}_1 : \operatorname{Spec} \Omega_1 \to U_{\overline{K}}$ be a geometric point of $U_{\overline{K}}$. By composition it induces geometric points of U_K, U and $U_{\overline{R}}$, which we again denote by $\bar{z}_1$, and it also induces a geometric point $\bar{\eta} : \operatorname{Spec} \Omega_1 \to \operatorname{Spec} R$ with image the generic point η of $\operatorname{Spec} R$. Similarly, let $\bar{z}_2 : \operatorname{Spec} \Omega_2 \to U_{\bar{k}}$ be a geometric point of $U_{\bar{k}}$. By composition this induces geometric points of U_k, U and $U_{\overline{R}}$, which we again denote by $\bar{z}_2$, and it also induces a geometric point $\bar{\pi} : \operatorname{Spec} \Omega_2 \to \operatorname{Spec} R$ with image the unique closed point of $\operatorname{Spec} R$ corresponding to the maximal ideal $\pi R \subset R$. As in section 1.1, it follows from functoriality of the fundamental group along with Theorem 1.4.10 that there is a commutative diagram:

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(U_{\overline{K}}, \bar{z}_1) \arrow[r] \arrow[d] & \pi_1(U_K, \bar{z}_1) \arrow[r] \arrow[d] & G_K \arrow[r] \arrow[d, two heads] & 1 \\
1 \arrow[r] & \pi_1(U_{\overline{R}}, \bar{z}_1) \arrow[r] \arrow[d, phantom, "\wr|"] & \pi_1(U, \bar{z}_1) \arrow[r] \arrow[d, phantom, "\wr|"] & \pi_1(\operatorname{Spec} R, \bar{\eta}) \arrow[r] \arrow[d, phantom, "\wr|"] & 1 \\
1 \arrow[r] & \pi_1(U_{\overline{R}}, \bar{z}_2) \arrow[r] & \pi_1(U, \bar{z}_2) \arrow[r] & \pi_1(\operatorname{Spec} R, \bar{\pi}) \arrow[r] & 1 \\
1 \arrow[r] & \pi_1(U_{\bar{k}}, \bar{z}_2) \arrow[r] \arrow[u] & \pi_1(U_k, \bar{z}_2) \arrow[r] \arrow[u] & G_k \arrow[r] \arrow[u, "\wr"] & 1
\end{tikzcd} \tag{3.4}$$

Lemma 3.2.2. (i) *The homomorphisms $\pi_1(U_{\overline{K}}, \bar{z}_1) \to \pi_1(U_{\overline{R}}, \bar{z}_1)$ and $\pi_1(U_K, \bar{z}_1) \to \pi_1(U, \bar{z}_1)$ are surjective.*

(ii) *The homomorphism $\pi_1(U_{\overline{K}}, \bar{z}_1) \to \pi_1(U_{\overline{R}}, \bar{z}_1)$ is an isomorphism when k has characteristic zero.*

Proof. The proof of (i) is identical to the proof of Lemma 3.1.2 (i). For (ii), we need to show that any finite morphism $Y_{\overline{K}} \to X_{\overline{K}}$ which is étale over $U_{\overline{K}}$ comes from a finite morphism $Y_{\overline{R}} \to X_{\overline{R}}$ which is étale over $U_{\overline{R}}$. The argument proceeds exactly as in the proof of Lemma 3.1.2 (ii), except that Zariski's Purity Theorem implies that the finite morphism $Y_{R'} \to X_{R'}$ can only be ramified over the points of $S_{R'}$. $\square$

Theorem 3.2.3. *Assume that k has characteristic zero. Then the homomorphism $\pi_1(U_{\bar{k}}, \bar{z}_2) \to \pi_1(U_{\overline{R}}, \bar{z}_2)$ is an isomorphism.*

Proof. To prove surjectivity it suffices to show that, for any finite morphism $Y \to X_{\overline{R}}$ étale over $U_{\overline{R}}$ with Y connected, $Y_{\bar{k}}$ is also connected. This is proved using the Stein factorisation of the proper morphism $Y \to \operatorname{Spec} \overline{R}$, exactly as in the proof of Theorem 3.1.4.

For injectivity we follow the formal patching method used in [Saï04, §1]. We need to show that a finite morphism $f_{\bar{k}} : \overline{Y} \to X_{\bar{k}}$ which is étale over $U_{\bar{k}}$ and possibly ramified at the points of $S_{\bar{k}}$ can be lifted to a finite morphism $f : Y \to X_{\overline{R}}$ which is étale over $U_{\overline{R}}$ and possibly ramified above the points of $S_{\overline{R}}$, such that $\overline{Y} = Y_{\bar{k}}$.

There exists a finite extension $l|k$ such that $f_{\bar{k}} : \overline{Y} \to X_{\bar{k}}$ descends to a finite morphism $f_l : Y_l \to X_l$ which is étale over U_l (this follows from [Sza09, Lemma 5.6.2] and Proposition 1.2.5). We may assume that f_l is ramified over S_l. Indeed, if it is ramified only over a subset $S' \subset S_l$, we may replace S_l with S' and U_l with $X_l - S'$ in the rest of the proof.

Since S_l consists of finitely many points, we may take a finite, unramified extension R' of R such that, denoting by k' the residue field of R' and K' its fraction field, k' contains l and the points of $S_{k'}$ are k'-rational. Since S is étale over R, $S_{R'}$ is étale over R', and thus the points of $S_{K'}$ are K'-rational and in bijection with the points of $S_{k'}$. Then $f_{k'} : Y_{k'} \to X_{k'}$ is étale over $U_{k'}$ and ramified over $S_{k'}$. We will show that $f_{k'}$ lifts to a finite morphism $f_{R'} : Y_{R'} \to X_{R'}$ which is étale over $U_{R'}$ and ramified over $S_{R'}$. Then $Y := Y_{R'} \times_{\operatorname{Spec} R'} \operatorname{Spec} \overline{R} \to X_{\overline{R}}$ is the desired morphism which lifts $f_{\bar{k}} : \overline{Y} \to X_{\bar{k}}$.

We can consider $f_{k'}$ to be composed of étale and ramified "parts". First, denoting $V_{k'} := f_{k'}^{-1}(U_{k'})$, the morphism $f_{k'}$ induces an étale cover $f_{k'}|_{V_{k'}} : V_{k'} \to U_{k'}$. Then, for each $x \in S_{k'}$, $f_{k'}$ induces a finite morphism $\hat{Y}_{k',x} \to \operatorname{Spec} \hat{\mathcal{O}}_{X_{k'},x}$, ramified at x, where $\hat{\mathcal{O}}_{X_{k'},x}$ is the completion of $\mathcal{O}_{X_{k'},x}$ with respect to its maximal ideal, and $\hat{Y}_{k',x}$ is the completion of $Y_{k'}$ along the closed subscheme defined by the preimage of x in $Y_{k'}$.

Let $\hat{X}_{R'}$ denote the completion of $X_{R'}$ along its closed fibre $X_{k'}$ (see [Har77, Ch. II, §9, p. 194] for a definition), and similarly, $\hat{U}_{R'}$ the completion of $U_{R'}$ along

$U_{k'}$. For each $x \in S_{k'}$, consider also the completion $\operatorname{Spf} \hat{\mathcal{O}}_{X_{R'},x}$ of $X_{R'}$ at the point x, where $\hat{\mathcal{O}}_{X_{R'},x}$ is the completion of $\mathcal{O}_{X_{R'},x}$ with respect to its maximal ideal and $\operatorname{Spf} \hat{\mathcal{O}}_{X_{R'},x}$ denotes its formal spectrum (see Notation). This can be considered a "formal neighbourhood" of x in $\hat{X}_{R'}$. To lift $f_{k'}$ to a finite morphism $f_{R'} : Y_{R'} \to X_{R'}$ we use a formal patching argument, wherein we find lifts of $V_{k'}$ and $\hat{Y}_{k',x}$ to $\hat{U}_{R'}$ and $\operatorname{Spf} \hat{\mathcal{O}}_{X_{R'},x}$ respectively, then "glue" them together.

Proposition 3.2.4. *With the above notation, we have the following.*

(i) *There exists a unique étale formal cover $\hat{V}_{R'} \to \hat{U}_{R'}$ which lifts the étale cover $f_{k'}|_{V_{k'}} : V_{k'} \to U_{k'}$.*

(ii) *For each $x \in S_{k'}$ there exists a finite morphism $\hat{Y}_{R',x} \to \operatorname{Spf} \hat{\mathcal{O}}_{X_{R'},x}$ which lifts the finite morphism $\hat{Y}_{k',x} \to \operatorname{Spec} \hat{\mathcal{O}}_{X_{k'},x}$ induced by $f_{k'}$.*

Proof. Statement (i) follows from [Gro67, Ch. IV, Théorème 18.1.2] and [Gro67, Ch. III, Théorème 5.1.4], exactly as in the proof of Theorem 3.1.4.

It remains to prove statement (ii). Since $X_{k'}$ is smooth, $\hat{\mathcal{O}}_{X_{k'},x}$ is a complete discrete valuation ring. Since x is k'-rational, we have an isomorphism $\hat{\mathcal{O}}_{X_{k'},x} \simeq k'[[t]]$, where t is a local parameter at x, so that x corresponds to the maximal ideal $(t) \subset k'[[t]]$ [Ser79, Ch. 2, §4]. Similarly, since the point $y \in S_{K'}$ specialising to x is K'-rational, $\hat{\mathcal{O}}_{X_{R'},x}$ is isomorphic to $R'[[T]]$, where T is a local parameter at x whose image modulo π is t, and the point y corresponds to the prime ideal $(T) \subset R'[[T]]$. Since the morphism $\hat{Y}_{k',x} \to \operatorname{Spec} \hat{\mathcal{O}}_{X_{k'},x}$ is finite and ramified at x, $\hat{Y}_{k',x}$ is isomorphic to $\operatorname{Spec} k'[[t]][z]/(z^n - t)$, for some n. To lift this to $\hat{\mathcal{O}}_{X_{R'},x}$ we need to lift the equation $z^n - t$ to a polynomial in $R'[[T]][Z]$ which is ramified at y, i.e. at the ideal (T). For this we can just take the same equation, $Z^n - T$, so we indeed obtain a finite morphism $\hat{Y}_{R',x} \simeq \operatorname{Spf} R'[[T]][Z]/(Z^n - T) \to \operatorname{Spf} \hat{\mathcal{O}}_{X_{R'},x}$ which is ramified at y and which lifts $\hat{Y}_{k',x} \to \operatorname{Spec} \hat{\mathcal{O}}_{X_{k'},x}$. □

The patching of the above morphisms is enabled by the following proposition (see [Saï04, Proposition 1.6]).

Proposition 3.2.5 (Formal patching)**.** *With notation as in Proposition 3.2.4, there exists a unique (up to isomorphism) finite morphism $f_{R'} : Y_{R'} \to X_{R'}$ which lifts $f_{k'} : Y_{k'} \to X_{k'}$ and which induces the étale formal cover $\hat{V}_{R'} \to \hat{U}_{R'}$, resp. the finite morphisms $\hat{Y}_{R',x} \to \operatorname{Spf} \hat{\mathcal{O}}_{X_{R'},x}$, when pulled back to $\hat{U}_{R'}$, resp. to $\operatorname{Spf} \hat{\mathcal{O}}_{X_{R'},x}$ for each $x \in S_{k'}$.*

Proof. To "glue" the morphisms $\hat{V}_{R'} \to \hat{U}_{R'}$ and $\hat{Y}_{R',x} \to \operatorname{Spf} \hat{\mathcal{O}}_{X_{R'},x}$ to give a finite morphism $Y_{R'} \to X_{R'}$ it suffices, by the formal GAGA theorem [Har03, Theorem 3.2.8], to check that they agree over the "overlap" of $\hat{U}_{R'}$ and $\operatorname{Spf} \hat{\mathcal{O}}_{X_{R'},x}$, which is the formal scheme $\operatorname{Spf}(\hat{\mathcal{O}}_{X_{R'},x})_{\mathfrak{p}}$, where $\mathfrak{p}$ is the minimal prime ideal of $\hat{\mathcal{O}}_{X_{R'},x}$ containing the uniformiser π and $(\hat{\mathcal{O}}_{X_{R'},x})_{\mathfrak{p}}$ is the localisation of $\hat{\mathcal{O}}_{X_{R'},x}$ at $\mathfrak{p}$.

The morphisms $\hat{V}_{R'} \to \hat{U}_{R'}$ and $\hat{Y}_{R',x} \to \operatorname{Spf} \hat{\mathcal{O}}_{X_{R'},x}$ induce, by pullback, morphisms $\hat{V}_{\mathfrak{p}} \to \operatorname{Spf}(\hat{\mathcal{O}}_{X_{R'},x})_{\mathfrak{p}}$ and $\hat{Y}_{\mathfrak{p}} \to \operatorname{Spf}(\hat{\mathcal{O}}_{X_{R'},x})_{\mathfrak{p}}$. These morphisms coincide over the closed fibre of $\operatorname{Spf}(\hat{\mathcal{O}}_{X_{R'},x})_{\mathfrak{p}}$, because they were both constructed from $f_{k'}$. Since both $\hat{V}_{\mathfrak{p}}$ and $\hat{Y}_{\mathfrak{p}}$ are étale over $\operatorname{Spf}(\hat{\mathcal{O}}_{X_{R'},x})_{\mathfrak{p}}$, and since $\operatorname{Spf}(\hat{\mathcal{O}}_{X_{R'},x})_{\mathfrak{p}}$ is local and complete, $\hat{V}_{\mathfrak{p}}$ and $\hat{Y}_{\mathfrak{p}}$ are indeed isomorphic. □

This concludes the proof of the Theorem. □

Corollary 3.2.6. *The homomorphism $\pi_1(U_k, \bar{z}_2) \to \pi_1(U, \bar{z}_2)$ is an isomorphism when k has characteristic zero.*

Proof. As in the proof of Corollary 3.1.5, this follows from the following commutative diagram.

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(U_{\overline{R}}, \bar{z}_2) \arrow[r] & \pi_1(U, \bar{z}_2) \arrow[r] & \pi_1(\operatorname{Spec} R, \bar{\pi}) \arrow[r] & 1 \\
1 \arrow[r] & \pi_1(U_{\bar{k}}, \bar{z}_2) \arrow[r] \arrow[u, "\wr"'] & \pi_1(U_k, \bar{z}_2) \arrow[r] \arrow[u] & G_k \arrow[r] \arrow[u, "\wr"'] & 1
\end{tikzcd}$$

□

Thus, under the assumption that k has characteristic zero, diagram 3.4 becomes:

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(U_{\overline{K}}, \bar{z}_1) \arrow[r] \arrow[d, "\wr"] & \pi_1(U_K, \bar{z}_1) \arrow[r] \arrow[d, two heads] & G_K \arrow[r] \arrow[d, two heads] & 1 \\
1 \arrow[r] & \pi_1(U_{\overline{R}}, \bar{z}_1) \arrow[r] \arrow[d, phantom, "\wr|"] & \pi_1(U, \bar{z}_1) \arrow[r] \arrow[d, phantom, "\wr|"] & \pi_1(\operatorname{Spec} R, \bar{\eta}) \arrow[r] \arrow[d, phantom, "\wr|"] & 1 \\
1 \arrow[r] & \pi_1(U_{\overline{R}}, \bar{z}_2) \arrow[r] & \pi_1(U, \bar{z}_2) \arrow[r] & \pi_1(\operatorname{Spec} R, \bar{\pi}) \arrow[r] & 1 \\
1 \arrow[r] & \pi_1(U_{\bar{k}}, \bar{z}_2) \arrow[r] \arrow[u, "\wr"] & \pi_1(U_k, \bar{z}_2) \arrow[r] \arrow[u, "\wr"] & G_k \arrow[r] \arrow[u, "\wr"] & 1
\end{tikzcd}$$

where the homomorphism $\pi_1(U_{\overline{K}}, \bar{z}_1) \to \pi_1(U_{\overline{R}}, \bar{z}_1)$ is an isomorphism by Lemma

3.2.2 (ii). Composing the vertical maps, we obtain surjective homomorphisms Sp_U and ρ and an isomorphism $\overline{\mathrm{Sp}}_U$ making the following diagram commutative.

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(U_{\overline{K}}, \bar{z}_1) \arrow[r] \arrow[d, "\overline{\mathrm{Sp}}_U"', "\wr"] & \pi_1(U_K, \bar{z}_1) \arrow[r] \arrow[d, two heads, "\mathrm{Sp}_U"'] & G_K \arrow[r] \arrow[d, two heads, "\rho"'] & 1 \\
1 \arrow[r] & \pi_1(U_{\bar{k}}, \bar{z}_2) \arrow[r] & \pi_1(U_k, \bar{z}_2) \arrow[r] & G_k \arrow[r] & 1
\end{tikzcd} \tag{3.5}$$

The homomorphisms $\overline{\mathrm{Sp}}_U$, Sp_U are called *specialisation homomorphisms of fundamental groups*, and they are defined only up to inner automorphism of $\pi_1(U_{\overline{R}}, \bar{z}_2)$, respectively $\pi_1(U, \bar{z}_2)$.

3.3 A specialisation homomorphism for absolute Galois groups

As in the previous section, let R be a complete discrete valuation ring with field of fractions K and residue field k. Fix an algebraic closure $\overline{K}$ of K. Denote by $\overline{R}$ the integral closure of R in $\overline{K}$, and by $\bar{k}$ the residue field of $\overline{R}$, which is an algebraic closure of k. Let X be a smooth, geometrically connected relative curve over R.

In this section we construct a homomorphism $G_{X_K} \to G_{X_k}$ between the absolute Galois groups of the generic and closed fibres of X. For this we will use the identity of Lemma 1.4.21, along with the specialisation homomorphism of fundamental groups of affine curves from the previous section. However, this method yields directly only a homomorphism from a quotient of G_{X_K}, for which we need the following definition.

Definition 3.3.1. Let C be a smooth, projective, geometrically connected curve over a field F, and let $\tilde{B}$ be an infinite set of closed points of C. Let $\bar{\xi}$ be a geometric point with image the generic point of C. We define the group $\pi_1(C - \tilde{B})$ to be the inverse limit

$$\pi_1(C - \tilde{B}) = \varprojlim_{B \subset \tilde{B} \text{ finite}} \pi_1(C - B, \bar{\xi})$$

where the limit is taken over the open subsets of C of the form $C - B$ (i.e. whose complements are finite subsets of $\tilde{B}$), ordered by inclusion.

Definition 3.3.2. A *universal pro-étale cover* $\tilde{C}_{\tilde{B}} \to C - \tilde{B}$ is an inverse system of finite morphisms $Y_i \to C$ such that:

(i) for each i, the branch locus of $Y_i \to C$ is contained in $\tilde{B}$;

(ii) for any finite morphism $Y \to C$ with branch locus contained in $\tilde{B}$ there is a morphism $Y_i \to Y$ for i sufficiently large.

A *point* $\tilde{c}$ of $\tilde{C}_{\tilde{B}}$ is a compatible system of points $y_i \in Y_i$.

Definition 3.3.3. Let $\tilde{C}_{\tilde{B}} \to C - \tilde{B}$ be a universal pro-étale cover, and let c be a point of C and $\tilde{c}$ a point of $\tilde{C}_{\tilde{B}}$ above c. The *decomposition group* $D_{\tilde{c}}$ of $\tilde{c}$ is the stabiliser of $\tilde{c}$ under the action of $\pi_1(C - \tilde{B})$. The *inertia group* $I_{\tilde{c}}$ is the kernel of the homomorphism $D_{\tilde{c}} \to G_{k(c)}$.

Remark 3.3.4. (i) The decomposition and inertia groups of Definition 3.3.3 satisfy analogous properties to those of Definitions 1.4.14 and 1.4.22. In particular, they may be written as inverse limits of decomposition and inertia subgroups of the fundamental groups of the open subsets of C of the form $C - B$, for finite subsets $B \subset \tilde{B}$ (see the paragraph after Definition 1.4.22).

(ii) While we omit the base point $\bar{\xi}$ in the notation $\pi_1(C - \tilde{B})$, it is implicit that this group comes with the choice of base point $\bar{\xi}$, with which $\pi_1(C - \tilde{B})$ is naturally a quotient of G_C (see Lemma 1.4.21).

As in Definition 3.3.1, we will consider infinite sets of closed points of the curves X_K and $X_{\overline{K}}$, the construction of which requires the following Lemma.

Lemma 3.3.5. *For each closed point x of X_k, there exists a closed point y of X_K specialising to x whose residue field is the unique unramified extension L of K whose valuation ring $\mathcal{O}_L$ has residue field $k(x)$.*

Proof. The closed point x is defined by a section $\operatorname{Spec} k(x) \to X_k$ (such a section is not unique - there are $[k(x) : k]$ such sections). By composition with the projection morphism $X_k \to X$, this defines a morphism $\operatorname{Spec} k(x) \to X_k \to X$. We will denote this composite morphism and its image in X both by x.

Denote by π the uniformiser of R and by π_L the uniformiser of $\mathcal{O}_L$ (where π maps to π_L under the inclusion $R \hookrightarrow \mathcal{O}_L$). Then, for each positive integer n, we have a composite morphism $\operatorname{Spec} \mathcal{O}_L / \pi_L^n \to \operatorname{Spec} \mathcal{O}_L \to \operatorname{Spec} R$ and, since $k(x) = \mathcal{O}_L / \pi_L$

by definition, a morphism $\operatorname{Spec} k(x) \to \operatorname{Spec} \mathcal{O}_L / \pi_L^n$. Thus for each n we have a commutative diagram:

$$\begin{tikzcd}
X \arrow[r] & \operatorname{Spec} R \\
& \operatorname{Spec} \mathcal{O}_L / \pi_L^n \arrow[u] \\
X_k \arrow[uu] & \operatorname{Spec} k(x) \arrow[l] \arrow[u] \arrow[uul, bend left, "x"]
\end{tikzcd}$$

The morphism $x : \operatorname{Spec} k(x) \to X$ extends to a morphism $x_n : \operatorname{Spec} \mathcal{O}_L / \pi_L^n \to X$. A proof of this statement may be found in [GR71, Exposé III, Théorème 3.1 (iii)], but we will explain the argument here.

Since $X \to \operatorname{Spec} R$ is a smooth morphism, there exists an étale morphism $U \to \mathbb{A}^1_R$ for some open neighbourhood U of x in X [Liu02, Corollary 6.2.11]. We therefore have a composite morphism $\operatorname{Spec} k(x) \to U \to \mathbb{A}^1_R$, which is defined by a ring homomorphism $R[T] \to k(x)$, and with a suitable choice of the parameter T we may assume that T maps to zero under this homomorphism. The morphism $\operatorname{Spec} k(x) \to \mathbb{A}^1_R$ therefore extends to a morphism $\operatorname{Spec} R \to \mathbb{A}^1_R$, defined by the ring homomorphism $R[T] \to R$ with $T \mapsto \pi$ (this homomorphism is clearly compatible with the projection to $k(x)$). For each n, we may compose this morphism with the morphism $\operatorname{Spec} \mathcal{O}_L / \pi_L^n \to \operatorname{Spec} R$ to obtain a morphism $\operatorname{Spec} \mathcal{O}_L / \pi_L^n \to \mathbb{A}^1_R$.

Thus we may take the fibre products $U \times_{\mathbb{A}^1_R} \operatorname{Spec} k(x)$ and $U \times_{\mathbb{A}^1_R} \operatorname{Spec} \mathcal{O}_L / \pi_L^n$, and we have a commutative diagram

$$\begin{tikzcd}
U \times_{\mathbb{A}^1_R} \operatorname{Spec} k(x) \arrow[r] \arrow[d] & U \times_{\mathbb{A}^1_R} \operatorname{Spec} \mathcal{O}_L / \pi_L^n \arrow[d] \\
\operatorname{Spec} k(x) \arrow[r] \arrow[u, bend right, "x"'] & \operatorname{Spec} \mathcal{O}_L / \pi_L^n
\end{tikzcd}$$

where the downward vertical morphisms are both étale (since $U \to \mathbb{A}^1_R$ is étale and étale morphisms are stable under base change), and the upper horizontal morphism and the upward left vertical morphism each come from the universal property of pullbacks - e.g. the morphism $x : \operatorname{Spec} k(x) \to U$ together with the identity morphism $\operatorname{Spec} k(x) \to \operatorname{Spec} k(x)$ uniquely determines a morphism $\operatorname{Spec} k(x) \to$

$U \times_{\mathbb{A}^1_R} \operatorname{Spec} k(x)$, which we also denote by x in the diagram.

Since $\operatorname{Spec} k(x)$ and $\operatorname{Spec} \mathcal{O}_L / \pi_L^n$ both have the same underlying topological space, the morphism $x : \operatorname{Spec} k(x) \to U \times_{\mathbb{A}^1_R} \operatorname{Spec} k(x)$ uniquely extends to a morphism $x_n : \operatorname{Spec} \mathcal{O}_L / \pi_L^n \to U \times_{\mathbb{A}^1_R} \operatorname{Spec} \mathcal{O}_L / \pi_L^n$ (see [GR71, Exposé I, Théorème 5.5]). Composing this with the projection and inclusion morphisms $U \times_{\mathbb{A}^1_R} \operatorname{Spec} \mathcal{O}_L / \pi_L^n \to U \hookrightarrow X$, we obtain a morphism $\operatorname{Spec} \mathcal{O}_L / \pi_L^n \to X$, which we also denote by x_n.

Since such a morphism exists for every positive integer n, in the projective limit over n we obtain a morphism $\hat{x} : \operatorname{Spf} \mathcal{O}_L \to \hat{X}$, where $\operatorname{Spf} \mathcal{O}_L$ is the formal spectrum of $\mathcal{O}_L$ and $\hat{X}$ is the completion of X along its closed fibre X_k (see Notation). Locally this is given by morphisms $\operatorname{Spf} \mathcal{O}_L \to \operatorname{Spf} \hat{A}$, where $\hat{A} := \varprojlim_n A/\pi_L^n$ is the completion with respect to π_L of a ring A such that $\operatorname{Spec} A$ is an affine open subset of X. Such a morphism is defined by a ring homomorphism $\hat{A} \to \mathcal{O}_L$, which may be composed with the natural inclusion $A \hookrightarrow \hat{A}$ to give a homomorphism $A \to \mathcal{O}_L$. This in turn defines a morphism $\operatorname{Spec} \mathcal{O}_L \to \operatorname{Spec} A$, and since such a morphism exists for every affine open subset of X we have a morphism $\operatorname{Spec} \mathcal{O}_L \to X$. The restriction of this morphism to $\operatorname{Spec} L$ defines a morphism $\operatorname{Spec} L \to X_K$, which defines the desired point y. $\square$

Remark 3.3.6. We can generalise the above Lemma as follows: for any finite extension M of K whose valuation ring has residue field $k(x)$, there exists an M-rational point of X_K specialising to x. Indeed, our proof only requires that the morphism $X \to \operatorname{Spec} R$ is smooth, and does not use the fact that $L|K$ is an unramified extension.

Definition 3.3.7. For each closed point x of X_k, fix a choice of closed point $y \in X_K^{\mathrm{cl}}$ of the generic fibre which specialises to x and whose residue field is the unique unramified extension of K whose valuation ring has residue field $k(x)$ (such a point exists by Lemma 3.3.5 above). We define $\tilde{S}$ to be the set of these chosen closed points $y \in X_K^{\mathrm{cl}}$. Thus, $\tilde{S}$ is a subset of X_K^{cl} in bijection with X_k^{cl}.

We denote by $\tilde{S}_{\overline{K}}$ the base change of $\tilde{S}$ to $\overline{K}$. Thus $\tilde{S}_{\overline{K}}$ is a subset of $X_{\overline{K}}^{\mathrm{cl}}$ in bijection with $X_{\bar{k}}^{\mathrm{cl}}$.

Let $B = \{y_1, \ldots, y_n\} \subset \tilde{S}$ be any finite subset of $\tilde{S}$, and denote by S the closure of B in X, that is the union $\overline{\{y_1\}} \cup \cdots \cup \overline{\{y_n\}} \subset X$. Then $S_K = B$, and, denoting the specialisation of y_i by x_i, S_k is the finite subset $\{x_1, \ldots, x_n\} \subset X_k^{\mathrm{cl}}$. Since the points y_i were chosen so that their residue fields are unramified extensions of K (see the above Definition), S is a divisor on X which is finite étale over R, by Lemma 3.2.1. Then the base change $S_{\overline{R}}$ of any such S to $\overline{R}$ is a divisor on $X_{\overline{R}}$ which is finite étale over $\overline{R}$, since finite and étale morphisms are stable under base change.

Theorem 3.3.8. *Assume k has characteristic zero. Then there exist surjective homomorphisms* Sp *and* ρ *and an isomorphism* $\overline{\mathrm{Sp}}$ *making the following diagram commutative.*

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(X_{\overline{K}} - \tilde{S}_{\overline{K}}) \arrow[r] \arrow[d, "\overline{\mathrm{Sp}}"', "\wr"] & \pi_1(X_K - \tilde{S}) \arrow[r] \arrow[d, two heads, "\mathrm{Sp}"'] & G_K \arrow[r] \arrow[d, two heads, "\rho"'] & 1 \\
1 \arrow[r] & G_{X_{\bar{k}}} \arrow[r] & G_{X_k} \arrow[r] & G_k \arrow[r] & 1
\end{tikzcd} \tag{3.6}$$

The homomorphisms $\overline{\mathrm{Sp}}$ and Sp will be referred to as *specialisation homomorphisms.* As in diagram 3.5, Sp, resp. $\overline{\mathrm{Sp}}$, is defined only up to inner automorphism of G_{X_k}, resp. $G_{X_{\bar{k}}}$.

Proof. Let $\bar{\xi}_1 : \operatorname{Spec}\Omega_1 \to X_{\overline{K}}$ be a geometric point with image the generic point of $X_{\overline{K}}$, and similarly let $\bar{\xi}_2 : \operatorname{Spec}\Omega_2 \to X_{\bar{k}}$ be a geometric point with image the generic point of $X_{\bar{k}}$. By composition these induce geometric points of X_K and X_k, which we also denote by $\bar{\xi}_1$ and $\bar{\xi}_2$ respectively.

Let $B \subset \tilde{S}$ be a finite subset of $\tilde{S}$, and let S denote its closure in X. Thus, S is a divisor on X which is finite étale over R such that $S_K = B$. Denote $U_K = X_K - S_K$ and $U_k = X_k - S_k$. By Corollary 3.2.6 and Lemma 3.2.2 (i), there exists a surjective specialisation homomorphism $\mathrm{Sp}_U : \pi_1(U_K, \bar{\xi}_1) \twoheadrightarrow \pi_1(U_k, \bar{\xi}_2)$. Since such a homomorphism exists for every finite subset $B \subset \tilde{S}$, we have a compatible system of surjective homomorphisms $\{\mathrm{Sp}_U\}$ parameterised by the finite subsets $B \subset \tilde{S}$, and taking the inverse limit of this system yields the surjective homomorphism $\mathrm{Sp} : \pi_1(X_K - \tilde{S}) \to G_{X_k}$.

Similarly, by Theorem 3.2.3 and Lemma 3.2.2 (ii), there is a compatible system of isomorphisms $\overline{\mathrm{Sp}_U} : \pi_1(U_{\overline{K}}, \bar{\xi}_1) \simeq \pi_1(U_{\bar{k}}, \bar{\xi}_2)$, parameterised by the finite subsets of $\tilde{S}_{\overline{K}}$. The inverse limit of this system yields the isomorphism $\overline{\mathrm{Sp}} : \pi_1(X_{\overline{K}} - \tilde{S}_{\overline{K}}) \simeq G_{X_{\bar{k}}}$. Thus we have the required homomorphisms in diagram 3.6, and this diagram is clearly commutative. □

Since the groups $\pi_1(X_{\overline{K}} - \tilde{S}_{\overline{K}})$ and $\pi_1(X_K - \tilde{S})$ are naturally quotients of $G_{X_{\overline{K}}}$

and G_{X_K}, respectively, we can expand diagram 3.6:

$$\begin{tikzcd}
1 \arrow[r] & G_{X_{\overline{K}}} \arrow[r] \arrow[d, two heads] & G_{X_K} \arrow[r] \arrow[d, two heads] & G_K \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \pi_1(X_{\overline{K}} - \tilde{S}_{\overline{K}}) \arrow[r] \arrow[d, two heads, "\overline{\mathrm{Sp}}"'] & \pi_1(X_K - \tilde{S}) \arrow[r] \arrow[d, two heads, "\mathrm{Sp}"'] & G_K \arrow[r] \arrow[d, two heads, "\rho"'] & 1 \\
1 \arrow[r] & G_{X_{\bar{k}}} \arrow[r] & G_{X_k} \arrow[r] & G_k \arrow[r] & 1
\end{tikzcd} \tag{3.7}$$

One could consider the composite homomorphism $G_{X_K} \to \pi_1(X_K - \tilde{S}) \to G_{X_k}$, and similarly for $G_{X_{\overline{K}}}$, to be a specialisation homomorphism for absolute Galois groups. However, we will reserve the label 'Sp' for the homomorphism in Theorem 3.3.8, since this will be important in the next section.

3.4 Ramification of sections

Let R be a complete discrete valuation ring with field of fractions K and residue field k. *We will assume throughout this section that k has characteristic zero.* Fix an algebraic closure $\overline{K}$ of K, and let $\bar{k}$ denote the algebraic closure of k in $\overline{K}$.

Let X be a smooth, geometrically connected relative curve over $\operatorname{Spec} R$ whose closed fibre X_k is *hyperbolic*. Let S be a divisor of X which is finite étale over R, and denote by $U = X - S$ the complement. Let $\bar{z}_1 : \operatorname{Spec} \Omega_1 \to U_{\overline{K}}$ and $\bar{z}_2 : \operatorname{Spec} \Omega_2 \to U_{\bar{k}}$ be geometric points, and denote the induced geometric points of U_K and U_k again by $\bar{z}_1$ and $\bar{z}_2$ respectively. Recall diagram 3.5 from the end of section 3.2:

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(U_{\overline{K}}, \bar{z}_1) \arrow[r] \arrow[d, "\overline{\mathrm{Sp}}_U"', "\wr"] & \pi_1(U_K, \bar{z}_1) \arrow[r] \arrow[d, two heads, "\mathrm{Sp}_U"'] & G_K \arrow[r] \arrow[d, two heads, "\rho"'] & 1 \\
1 \arrow[r] & \pi_1(U_{\bar{k}}, \bar{z}_2) \arrow[r] & \pi_1(U_k, \bar{z}_2) \arrow[r] & G_k \arrow[r] & 1
\end{tikzcd} \tag{3.8}$$

Fix universal pro-étale covers $\tilde{U}_K \to U_K$ and $\tilde{U}_k \to U_k$, and let $\tilde{X}_{U_K}$ denote the normalisation of X_K in $\tilde{U}_K$, and likewise $\tilde{X}_{U_k}$ the normalisation of X_k in $\tilde{U}_k$ (see Definitions 1.4.12 and 1.4.13).

Lemma 3.4.1. *Let y be a closed point of X_K, $\tilde{y}$ a point of $\tilde{X}_{U_K}$ above y and $x \in X_k$ the specialisation of y. Then the image of the decomposition subgroup $D_{\tilde{y}} \subset$*

$\pi_1(U_K, \bar{z}_1)$ under Sp_U is contained in a decomposition subgroup $D_{\tilde{x}} \subset \pi_1(U_k, \bar{z}_2)$ for some $\tilde{x}$ in $\tilde{X}_{U_k}$ above x.

Proof. Let $\overline{V} \to U_k$ be an étale cover in the system of étale covers defining $\tilde{U}_k$, and let $\overline{Y} \to X_k$ denote the normalisation of X_k in $\overline{V}$. From the proof of Theorem 3.2.3, there exists a finite extension $K'|K$ such that, denoting by R' the integral closure of R in K' and by k' the residue field of R', the finite morphism $\overline{Y}_{k'} \to X_{k'}$ lifts to a finite morphism $\overline{Y}_{R'} \to X_{R'}$ which is étale over $U_{R'}$ and possibly ramified over the points of $S_{R'}$. The base change $\overline{Y}_{K'} \to X_{K'}$ is a finite morphism which is étale over $U_{K'}$ and possibly ramfied over $S_{K'}$.

Let $(V_i \to U_K)_i$ denote the inverse system of étale covers defining $\tilde{U}_K$, and for each i, let Y_i denote the normalisation of X_K in V_i. The point $\tilde{y} \in \tilde{X}_{U_K}$ is defined by a compatible system of points $(y_i) \in (Y_i)$. For some i sufficiently large, there exists a morphism $Y_i \to \overline{Y}_{K'}$. Let $\bar{y}$ denote the image of y_i in $\overline{Y}_{K'}$, and denote the image of $\bar{y}$ in $\overline{Y}_{R'}$ again by $\bar{y}$. Let H denote the image of $D_{\tilde{y}}$ in $\pi_1(U_{R'}, \bar{z}_1)$. Then H acts on $\overline{Y}_{R'}$, and since $D_{\tilde{y}}$ fixes $\bar{y} \in \overline{Y}_{K'}$, H fixes $\bar{y} \in \overline{Y}_{R'}$. Let C denote the closed subscheme $\overline{\{\bar{y}\}} \subset \overline{Y}_{R'}$, which is proper over $\operatorname{Spec} R'$ since it is the composition of a closed immersion $C \hookrightarrow \overline{Y}_{R'}$, a finite morphism $\overline{Y}_{R'} \to X_{R'}$ and a proper morphism $X_{R'} \to \operatorname{Spec} R'$. The action of H on C restricts to a morphism $\operatorname{Spec} k(C) \to C$. Denote by $\bar{x}$ the unique closed point of C, which is the specialisation of $\bar{y}$ to $\overline{Y}_{k'}$. Since $\mathcal{O}_{C,\bar{x}}$ is a discrete valuation ring, the valuative criterion of properness [Liu02, Corollary 3.3.26] implies that $\operatorname{Spec} k(C) \to C$ extends *uniquely* to a morphism $\operatorname{Spec} \mathcal{O}_{C,\bar{x}} \simeq C \to C$. This morphism is the action of H on C, and since it already fixes $\bar{y}$ it must be the identity morphism.

Thus H fixes $\bar{x} \in \overline{Y}_{k'}$, and since this holds for every étale cover in the system of étale covers defining $\tilde{U}_k$, the images of these $\bar{x}$ in their respective $\overline{Y}$ define a point $\tilde{x}$ of $\tilde{X}_{U_k}$ above x which is fixed by $\mathrm{Sp}_U(D_{\tilde{y}})$, that is, $\mathrm{Sp}_U(D_{\tilde{y}}) \subset D_{\tilde{x}}$. Note that, since Sp_U is only defined up to conjugation by $\pi_1(U, \bar{z}_2) \simeq \pi_1(U_k, \bar{z}_2)$, the image $\mathrm{Sp}_U(D_{\tilde{y}})$ may be contained in any of the conjugates $D_{\sigma\tilde{x}}$ for any $\sigma \in \pi_1(U_k, \bar{z}_2)$. □

Recall that the kernel of the projection $\rho : G_K \twoheadrightarrow G_k$ is the inertia group I_K associated to the discrete valuation on K.

Lemma 3.4.2. (i) *The projection $\pi_1(U_K, \bar{z}_1) \twoheadrightarrow G_K$ restricts to an isomorphism $\ker(\mathrm{Sp}_U) \simeq I_K$.*

(ii) *The right square in diagram 3.8 is a pullback square.*

Proof. The isomorphism $\ker(\mathrm{Sp}) \simeq I_K$ follows from a simple diagram chase, and (ii) follows easily from (i). □

Let x be a closed point of X_k which is not contained in S_k. Let $y \in X_K$ be a point specialising to x such that the residue field L of y is the unique unramified extension of K whose ring of integers $\mathcal{O}_L$ has residue field $k(x)$. Such a point exists by Lemma 3.3.5, and, moreover, y is not contained in S_K since $x \notin S_k$ and S is finite étale over R.

Let $\tilde{y}$ be a point of $\tilde{U}_K$ above y. By Lemma 3.4.1, the image of the decomposition group $D_{\tilde{y}}$ under Sp_U is contained in $D_{\tilde{x}}$ for some $\tilde{x}$ in $\tilde{U}_k$ above x. By Proposition 1.4.16 we have isomorphisms $D_{\tilde{y}} \simeq G_L$ and $D_{\tilde{x}} \simeq G_{k(x)}$. Together with commutativity of diagram 3.8, this implies that the image of $D_{\tilde{y}}$ under Sp_U is equal to $D_{\tilde{x}}$. Thus we have a diagram:

$$\begin{tikzcd} D_{\tilde{y}} \arrow[r, "\sim"] \arrow[d, two heads, "\mathrm{Sp}_U"'] & G_L \arrow[d, two heads, "\rho|_{G_L}"] \\ D_{\tilde{x}} \arrow[r, "\sim"] & G_{k(x)} \end{tikzcd} \tag{3.9}$$

Suppose now that $x \in S_k$. There are infinitely many points of X_K specialising to x, but since S is finite étale over R, there is exactly one point of S_K specialising to x, and its residue field is the unique unramified extension of K whose ring of integers $\mathcal{O}_L$ has residue field $k(x)$.

Lemma 3.4.3. *Let $x \in S_k$, and let y' be the unique point of S_K which specialises to x. Let L denote the residue field of y', which is the unique unramified extension of K whose ring of integers $\mathcal{O}_L$ has residue field $k(x)$. Let $\tilde{y}'$ be a point of $\tilde{X}_{U_K}$ above y'. For some $\tilde{x}$ in $\tilde{X}_{U_k}$ above x, we have the following commutative diagram:*

$$\begin{tikzcd} 1 \arrow[r] & I_{\tilde{y}'} \arrow[r] \arrow[d, "\overline{\mathrm{Sp}}_U"', "\wr"] & D_{\tilde{y}'} \arrow[r] \arrow[d, two heads, "\mathrm{Sp}_U"'] & G_L \arrow[r] \arrow[d, two heads, "\rho"'] & 1 \\ 1 \arrow[r] & I_{\tilde{x}} \arrow[r] & D_{\tilde{x}} \arrow[r] & G_{k(x)} \arrow[r] & 1 \end{tikzcd}$$

Moreover, the right square in this diagram is a pullback square, that is, $D_{\tilde{y}'}$ is the pullback of $D_{\tilde{x}}$ to G_L.

Proof. By Lemma 1.4.18, the image of $D_{\tilde{y}'}$ under Sp_U is contained in $D_{\tilde{x}}$ for some $\tilde{x}$ in

$\tilde{X}_{U_k}$ above x. Since $\overline{\mathrm{Sp}}_U$ is an isomorphism, this means the inertia subgroup $I_{\tilde{y}'} \subset D_{\tilde{y}'}$ maps isomorphically to $I_{\tilde{x}} \subset \pi_1(U_{\bar{k}}, \bar{z}_2)$. Commutativity of diagram 3.8 then implies that $D_{\tilde{y}'}$ maps surjectively onto $D_{\tilde{x}}$, whence the diagram of the Lemma. As in the proof of Lemma 3.4.2, the right square in this diagram is a pullback square. □

Lemma 3.4.4. *With the notation of Lemma 3.4.3, for any closed point y of U_K specialising to x and any point $\tilde{y}$ of $\tilde{U}_K$ above y, $D_{\tilde{y}}$ is contained in $D_{\tilde{y}'}$ for some $\tilde{y}'$ in $\tilde{X}_{U_K}$ above y'.*

Proof. Let L' denote the residue field of y, which is a finite extension of L. Then y is defined by a (non-unique) section $\operatorname{Spec} L' \to U_K$, which induces, by functoriality, a conjugacy class of sections $s_y : G'_L \to \pi_1(U_K, \bar{z}_1)$, where each such section s_y has image the decomposition group $D_{\tilde{y}}$ of some $\tilde{y}$ in $\tilde{U}_K$ above y (see Proposition 1.5.6 and the discussion after it). Writing $i_{\tilde{y}} : D_{\tilde{y}} \hookrightarrow \pi_1(U_K, \bar{z}_1)$ for the inclusion of $D_{\tilde{y}}$ into $\pi_1(U_K, \bar{z}_1)$ and φ_s for the composite $\varphi_s := \mathrm{Sp}_U \circ s_y = \mathrm{Sp}_U \circ i_{\tilde{y}} \circ s_y : G_{L'} \to \pi_1(U_k, \bar{z}_2)$, we have the following commutative diagram:

$$\begin{tikzcd}
D_{\tilde{y}} \arrow[d, hook, "i_{\tilde{y}}"'] & G_{L'} \arrow[l, "s_y"'] \arrow[d, hook] \arrow[ddl, bend right, "\varphi_s"'] \\
\pi_1(U_K, \bar{z}_1) \arrow[r] \arrow[d, two heads, "\mathrm{Sp}_U"'] & G_K \arrow[d, two heads, "\rho"] \\
\pi_1(U_k, \bar{z}_2) \arrow[r, two heads] & G_k
\end{tikzcd} \tag{3.10}$$

Since we have homomorphisms $\varphi_s : G_{L'} \to \pi_1(U_k, \bar{z}_2)$ and $G_{L'} \hookrightarrow G_K$, by Lemma 3.4.2 (ii) the pullback property implies there is a unique homomorphism $G_{L'} \to \pi_1(U_K, \bar{z}_1)$ commuting with all the homomorphisms in the above diagram. Thus, the composite $i_{\tilde{y}} \circ s_y$ is the unique such homomorphism.

By Lemma 3.4.1, the image of $D_{\tilde{y}}$ under Sp_U is contained in $D_{\tilde{x}}$ for some $\tilde{x}$ in $\tilde{X}_{U_k}$ above x. Thus we have homomorphisms $\varphi_s := \mathrm{Sp}_U \circ s_y : G_{L'} \to D_{\tilde{x}}$ and $G_{L'} \hookrightarrow G_L$

such that the following diagram is commutative:

$$\begin{tikzcd}
D_{\tilde{y}} \arrow[dd, two heads, "\mathrm{Sp}_U"'] & G_{L'} \arrow[l, "s_y"'] \arrow[ddl, bend right, "\varphi_s"] \arrow[dr, hook] & \\
 & & G_L \arrow[d, two heads, "\rho"'] \\
D_{\tilde{x}} \arrow[rr, two heads] & & G_{k(x)}
\end{tikzcd}$$

There exists some $\tilde{y}'$ in $\tilde{X}_{U_K}$ above y' such that the image of $D_{\tilde{y}'}$ under Sp_U is contained in $D_{\tilde{x}}$. Indeed, choose any $\tilde{w}$ in $\tilde{X}_{U_K}$ above y', and suppose that $\mathrm{Sp}_U(D_{\tilde{w}}) \subset D_{\sigma\tilde{x}}$ for some $\sigma \in \pi_1(U_k, \bar{z}_2)$. Then we may take any $\tau \in \pi_1(U_K, \bar{z}_1)$ with $\mathrm{Sp}_U(\tau) = \sigma$, and choose $\tilde{y}' = \tau^{-1}\tilde{w}$. By Lemma 3.4.3, the pullback property then implies the existence of a unique homomorphism $G_{L'} \to D_{\tilde{y}'}$ making the following diagram commutative:

$$\begin{tikzcd}
D_{\tilde{y}} \arrow[dd, two heads, "\mathrm{Sp}_U"'] & G_{L'} \arrow[l, "s_y"'] \arrow[ddl, bend right, "\varphi_s"] \arrow[d, "\exists!"'] \arrow[dr, hook] & \\
 & D_{\tilde{y}'} \arrow[r] \arrow[dl, two heads, "\mathrm{Sp}_U"] & G_L \arrow[d, two heads, "\rho"'] \\
D_{\tilde{x}} \arrow[rr, two heads] & & G_{k(x)}
\end{tikzcd}$$

The composite $G_{L'} \to D_{\tilde{y}'} \hookrightarrow \pi_1(U_K, \bar{z}_1)$ must then commute with all the homomorphisms in diagram 3.10. By the above discussion, it must therefore coincide with the composite $i_{\tilde{y}} \circ s_y : G_{L'} \to D_{\tilde{y}} \hookrightarrow \pi_1(U_K, \bar{z}_1)$, which implies that $s_y(G_{L'}) = D_{\tilde{y}} \subset D_{\tilde{y}'}$. □

Recall from Definition 1.5.7 that a *cuspidal* section of $\pi_1(U_K, \bar{z}_1)$, resp. of $\pi_1(U_k, \bar{x}_2)$, is a section which factors through the decomposition group of a point of $\tilde{X}_{U_K}$ above a point of S_K, resp. a point of $\tilde{X}_{U_k}$ above a point of S_k. The above Lemma immediately implies the following.

Corollary 3.4.5. *Let x be a k-rational point of X_k which is contained in S_k, and let y be a K-rational point of X_K specialising to x. Then a section $s : G_K \to \pi_1(U_K, \bar{z}_1)$ with image contained in $D_{\tilde{y}}$ for some $\tilde{y}$ in $\tilde{X}_{U_K}$ above y is cuspidal, even if $y \notin S_K$.*

Proof. By Lemma 3.4.4, $s(G_K) \subset D_{\tilde{y}'}$ for some $\tilde{y}'$ in $\tilde{X}_{U_K}$ above the unique point y' of S_K specialising to x. $\square$

Since $\pi_1(U_K, \bar{z}_1)$ is the pullback of $\pi_1(U_k, \bar{z}_2)$ to G_K (see Lemma 3.4.2), any element of $\pi_1(U_K, \bar{z}_1)$ can be written as a pair (τ, σ), where $\tau \in \pi_1(U_k, \bar{z}_2)$ and $\sigma \in G_K$, such that τ and σ both map to the same element of G_k. Therefore, any section $\bar{s} : G_k \to \pi_1(U_k, \bar{z}_2)$ pulls back to a section $s : G_K \to \pi_1(U_K, \bar{z}_1)$ defined by $s(\sigma) = (\bar{s}(\rho(\sigma)), \sigma)$ for any $\sigma \in G_K$.

However, not every section of $\pi_1(U_K, \bar{z}_1)$ induces a section of $\pi_1(U_k, \bar{z}_2)$. Given any section $s : G_K \to \pi_1(U_K, \bar{z}_1)$, let us define a homomorphism $\varphi_s : G_K \to \pi_1(U_k, \bar{z}_2)$ by the composition $\varphi_s := \mathrm{Sp}_U \circ s$.

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(U_{\overline{K}}, \bar{z}_1) \arrow[r] \arrow[d, "\overline{\mathrm{Sp}}_U"', "\wr"] & \pi_1(U_K, \bar{z}_1) \arrow[r] \arrow[d, two heads, "\mathrm{Sp}_U"'] & G_K \arrow[r] \arrow[l, bend right, "s"'] \arrow[dl, bend right, "\varphi_s"'] \arrow[d, two heads, "\rho"] & 1 \\
1 \arrow[r] & \pi_1(U_{\bar{k}}, \bar{z}_2) \arrow[r] & \pi_1(U_k, \bar{z}_2) \arrow[r] & G_k \arrow[r] & 1
\end{tikzcd} \tag{3.11}$$

Definition 3.4.6. We say the above section s is *unramified* if $\varphi_s(I_K) = 1$. Otherwise we say s is *ramified.*

An unramified section $s : G_K \to \pi_1(U_K, \bar{z}_1)$ naturally induces a section $\bar{s} : G_k \to \pi_1(U_k, \bar{z}_1)$ defined by $\bar{s}(\sigma I_K) = \varphi_s(\sigma)$ for every $\sigma \in G_K$, where σI_K is the coset of σ modulo I_K. This induced section will be called the *specialisation of* s.

For an unramified section s of $\pi_1(U_K, \bar{z}_1)$, we will denote its specialisation by $\bar{s}$.

One can easily see that, for unramified sections, the operations of specialisation and pullback are inverse to each other. That is, if $s : G_K \to \pi_1(U_K, \bar{z}_1)$ is unramified, then the pullback of $\bar{s}$ to G_K is precisely s.

Lemma 3.4.7. *Let y be a K-rational point of U_K such that the specialisation x of y is (k-rational and) contained in U_k, and let $s : G_K \to \pi_1(U_K, \bar{z}_1)$ be a section with $s(G_K) = D_{\tilde{y}}$ for some $\tilde{y}$ in $\tilde{U}_K$ above y. Then s is unramified, and $\bar{s}(G_k) = D_{\tilde{x}}$ for some $\tilde{x}$ in $\tilde{U}_k$ above x.*

Proof. Regarding y as a section $y : \operatorname{Spec} K \to X_K$, it induces, by the valuative criterion of properness [Liu02, Theorem 3.3.25], a section $y_R : \operatorname{Spec} R \to X$, which pulls back to a section $\bar{y} : \operatorname{Spec} k \to X_k$ with image the specialisation x of y. Since $y \in U_K$ and $x \in U_k$, these sections have images in U_K, U and U_k respectively.

Since $s(G_K) = D_{\tilde{y}}$, the section s arises from y by functoriality of the fundamental group (see Proposition 1.5.6). Similarly, y_R and $\bar{y}$ give rise to sections $s_R : \pi_1(\operatorname{Spec} R, \bar{\eta}) \to \pi_1(U, \bar{z}_1)$ and $\bar{s} : G_x \to \pi_1(U_k, \bar{z}_2)$. Since the specialisation homomorphism Sp_U also comes from functoriality (see diagram 3.4), the image $\bar{s}(G_k)$ of the section $\bar{s}$ must coincide with $\varphi_s(G_K)$. Thus $\varphi_s(I_K)$ must be trivial and $\bar{s}$ is induced by s via Sp_U as in Definition 3.4.6, that is, s is unramified and $\bar{s}$ is its specialisation. Since $\bar{s}$ arises from $\bar{y}$, we must have $\bar{s}(G_k) \subset D_{\tilde{x}}$ for some $\tilde{x}$ in $\tilde{U}_k$ above x. □

Proposition 3.4.8. *Assume that k satisfies condition (ii) in Definition 2.3.1. Let $s : G_K \to \pi_1(U_K, \bar{z}_1)$ be a section, and denote $\varphi_s := \mathrm{Sp}_U \circ s$ as above. If $\varphi_s(I_K)$ is non-trivial then it is contained in the inertia group $I_{\tilde{x}}$ of a unique point $\tilde{x}$ of $\tilde{X}_{U_k}$ above a point of S_k.*

Proof. For simplicity, let us write $H := \varphi_s(I_K)$. If H is non-trivial then, by commutativity of diagram 3.11, it must be contained in $\pi_1(U_{\bar{k}}, \bar{z}_2)$, and it is a procyclic subgroup of $\pi_1(U_{\bar{k}}, \bar{z}_2)$ because $I_K \simeq \hat{\mathbb{Z}}(1)$ is procyclic. Since k satisfies condition (ii) of Definition 2.3.1, the image of H under the composite homomorphism

$$\pi_1(U_{\bar{k}}, \bar{z}_2) \twoheadrightarrow \pi_1(X_{\bar{k}}, \bar{z}_2) \twoheadrightarrow \pi_1(X_{\bar{k}}, \bar{z}_2)^{\mathrm{ab}} \tag{3.12}$$

is trivial. The Proposition then follows from [HM11, Lemma 1.6], but we will reiterate the proof here, since this result is of central importance for the rest of this chapter.

Choose a universal pro-étale cover $\tilde{U}_{\bar{k}} \to U_{\bar{k}}$, and denote by $\tilde{X}_{U_{\bar{k}}}$ the normalisation of $X_{\bar{k}}$ in $\tilde{U}_{\bar{k}}$. Let N be any open normal subgroup of $\pi_1(U_{\bar{k}}, \bar{z}_2)$, and denote by $U_N \to U_{\bar{k}}$ the corresponding étale cover, X_N the normalisation of $X_{\bar{k}}$ in U_N (see Definition 1.4.13) and S_N the complement $X_N - U_N$. The product $H.N$ is also an open subgroup of $\pi_1(U_{\bar{k}}, \bar{z}_2)$, and thus corresponds to an étale cover $U_{H.N} \to U_{\bar{k}}$. We similarly denote by $X_{H.N}$ the normalisation of $X_{\bar{k}}$ in $U_{H.N}$ and by $S_{H.N}$ the complement $X_{H.N} - U_{H.N}$. Since H is procyclic, the quotient $H.N/N$ (the automorphism group of the étale cover $U_N \to U_{H.N}$) is finite and cyclic, hence we have a natural composite homomorphism

$$\bigoplus_{\substack{\tilde{x}\in\tilde{X}_{U_{\bar{k}}}\\ \text{above } S_{H.N}}} I_{\tilde{x}|x_{H.N}} \hookrightarrow H.N \twoheadrightarrow H.N^{\mathrm{ab}} \twoheadrightarrow H.N/N \tag{3.13}$$

where the direct sum is taken over all points $\tilde{x}$ of $\tilde{X}_{U_{\bar{k}}}$ above points of $S_{H.N}$, $I_{\tilde{x}|x_{H.N}}$ denotes the inertia subgroup of $H.N$ corresponding to the point $\tilde{x}$ of $\tilde{X}_{U_{\bar{k}}}$ above $x_{H.N} \in S_{H.N}$, and $H.N^{\mathrm{ab}}$ denotes the maximal abelian quotient of $H.N$.

Since $H.N/N$ is generated by the image of H, it must become trivial when we take the quotient by the image of $\oplus I_{\tilde{x}|x_{H.N}}$ in $H.N/N$ via the composite homomorphism 3.13 (since the image of H under the composite 3.12 is trivial). Therefore the composite 3.13 is surjective. Moreover, since $H.N/N$ is cyclic there must exist some $\tilde{x} \in \tilde{X}_{U_{\bar{k}}}$ above some $x_{H.N} \in S_{H.N}$ such that the composite homomorphism

$$I_{\tilde{x}|x_{H.N}} \hookrightarrow H.N \twoheadrightarrow H.N^{\mathrm{ab}} \twoheadrightarrow H.N/N \tag{3.14}$$

is surjective. That is, there exists a point $\tilde{x} \in \tilde{X}_{U_{\bar{k}}}$ above some $x_{H.N} \in S_{H.N}$ such that, denoting by x_N the image of $\tilde{x}$ in S_N, the finite morphism $X_N \to X_{H.N}$ is totally ramified at x_N. This means that $H.N/N$ has as a quotient the finite cyclic inertia group I_{x_N} (see Definition 1.3.7), hence the action of H on X_N must fix x_N. Denoting by $T_N \subset S_N$ the set of points of S_N which are fixed by the action of H on X_N, we therefore have that T_N is non-empty for each N, and since it is finite we conclude that $\varprojlim_N T_N$ is non-empty. An element $\tilde{x} \in \varprojlim_N T_N$ corresponds to a point of $\tilde{X}_{U_{\bar{k}}}$ above a point of $S_{\bar{k}}$, i.e. a compatible system of points $x_N \in S_N$ such that, for every N, the action of H on X_N fixes x_N. Therefore, H is contained in the inertia subgroup $I_{\tilde{x}} \subset \pi_1(U_{\bar{k}}, \bar{z}_2)$, and Proposition 1.4.17 (i) implies that this is the unique such inertia group. □

Remark 3.4.9. Under the hypotheses of the above Proposition, if $S = \emptyset$, so that $U = X$ is proper over $\operatorname{Spec} R$, then any section $s : G_K \to \pi_1(U_K, \bar{z}_1)$ is necessarily unramified. Indeed, $\pi_1(U_k, \bar{z}_2)$ contains no nontrivial inertia subgroups, so by the Proposition $\varphi_s(I_K)$ must be trivial.

Let $y \in X(K)$ be a K-rational point which specialises to a (k-rational) point x of S_k. Suppose that we are given a section $s : G_K \to \pi_1(U_K, \bar{z}_1)$ whose image is contained in $D_{\tilde{y}}$ for some $\tilde{y}$ in $\tilde{X}_{U_K}$ above y. Then s is a cuspidal section, by Lemma

3.4.4 and Corollary 3.4.5. If s is unramified, then necessarily $\bar{s}(G_k) \subset D_{\tilde{x}}$ for some $\tilde{x}$ in $\tilde{X}_{U_k}$ above x. Conversely, we have the following.

Lemma 3.4.10. *Let $s : G_K \to \pi_1(U_K, \bar{z}_1)$ be an unramified section, and suppose $\bar{s}(G_k) \subset D_{\tilde{x}}$ for some $x \in S(k)$ and some $\tilde{x}$ in $\tilde{X}_{U_k}$ above x. Then $s(G_K) \subset D_{\tilde{y}'}$ for some $\tilde{y}'$ in $\tilde{X}_{U_K}$ above the unique (K-rational) point $y' \in S(K)$ specialising to x.*

Proof. There exists some $\tilde{y}'$ in $\tilde{X}_{U_K}$ above y' such that the image of $D_{\tilde{y}'} \subset \pi_1(U_K, \bar{z}_1)$ under Sp_U is contained in $D_{\tilde{x}}$, and this $\tilde{y}'$ is unique since y' is K-rational and $\overline{\mathrm{Sp}}_U$ is an isomorphism (see the proof of Lemma 3.4.4). The pullback of $\bar{s}$ to G_K is exactly s (see Definition 3.4.6 and the paragraphs before and after it), so $s(G_K)$ must be contained in the pullback of $D_{\tilde{x}}$ to G_K, which is $D_{\tilde{y}'}$ by Lemma 3.4.3. □

A cuspidal section may still be ramified. The following Lemma shows that, with a suitable condition on k, the phenomenon of ramification is exclusive to the cuspidal sections.

Lemma 3.4.11. *Assume that k satisfies condition (ii) in Definition 2.3.1, and let $s : G_K \to \pi_1(U_K, \bar{z}_1)$ be a section and $\varphi_s := \mathrm{Sp}_U \circ s$. If s is ramified then $\varphi_s(G_K) \subset D_{\tilde{x}}$ for some k-rational point x of S_k and some $\tilde{x}$ in $\tilde{X}_{U_k}$ above x, and $s(G_K) \subset D_{\tilde{y}}$ where $\tilde{y}$ is a point of $\tilde{X}_{U_K}$ above the unique K-rational point of S_K specialising to x. In particular, s is cuspidal.*

Moreover, the k-rational point x is unique if k also satisfies condition (iii) (a) of Definition 2.3.1.

Proof. If $\varphi_s(I_K)$ is non-trivial then, by Proposition 3.4.8, it must be contained in a unique inertia group $I_{\tilde{x}}$ for some $x \in S_k$ and some $\tilde{x} \in \tilde{X}_{U_k}$ above x. Since $\varphi_s(G_K)$ normalises $\varphi_s(I_K)$, for some $\sigma \in G_K$ we have $\varphi_s(I_K) = \varphi_s(\sigma)^{-1} \cdot \varphi_s(I_K) \cdot \varphi_s(\sigma) \subseteq \varphi_s(\sigma)^{-1} \cdot I_{\tilde{x}} \cdot \varphi_s(\sigma) = I_{\varphi_s(\sigma)\cdot\tilde{x}}$. But $\varphi_s(I_K)$ is contained in a unique inertia group, so $\varphi_s(\sigma) \cdot \tilde{x} = \tilde{x}$ and $\varphi_s(G_K)$ fixes $\tilde{x}$, i.e. $\varphi_s(G_K) \subseteq D_{\tilde{x}}$. Moreover, x is necessarily a k-rational point of S_k since, by commutativity of diagram 3.11, $\varphi_s(G_K)$ maps surjectively onto G_k. Then Lemma 3.4.3 implies that $s(G_K) \subseteq D_{\tilde{y}}$ for some $\tilde{y}$ in $\tilde{X}_{U_K}$ above the unique K-rational point y of S_K specialising to x.

It remains to prove the uniqueness of x under condition (iii) (a) of Definition 2.3.1. Suppose there exists another k-rational point $x' \in X_k$ such that $\varphi_s(G_K) \subset D_{\tilde{x}'}$ for some $\tilde{x}'$ in $\tilde{X}_{U_k}$ above x'. The section s induces a section $s_X : G_K \to \pi_1(X_K, \bar{z}_1)$ of the étale fundamental group of the projective curve X_K, which is unramified (see Remark 3.4.9) and thus induces a section $\bar{s}_X : G_k \to \pi_1(X_k, \bar{z}_2)$. Choosing a universal pro-étale cover $\tilde{X} \to X$, Lemma 1.4.18 implies that $\bar{s}_X(G_k)$ is contained in $D_{\tilde{w}}$ and in $D_{\tilde{w}'}$ for some $\tilde{w}$, resp. $\tilde{w}'$ in $\tilde{X}$ above x, resp. x'. Thus $\bar{s}_X$ is conjugate to two sections $\bar{s}_x$, $\bar{s}_{x'} : G_k \to \pi_1(X_k, \bar{z}_2)$ induced from the elements $x, x' \in X_k(k)$ by functoriality of the fundamental group. Hence, $\bar{s}_x$ is conjugate to $\bar{s}_{x'}$, so Proposition 1.6.8 implies that $x = x'$. $\square$

Remark 3.4.12. The use of Proposition 1.6.8 in the above proof relies on the existence of a k-rational point of X_k, in order to identify the maximal abelian quotient $\pi_1(X_{\bar{k}}, \bar{z}_2)$ of the geometric fundamental group of X_k with the Tate module of the Jacobian of $X_{\bar{k}}$. In this case, the fact that s is ramified implies existence of a k-rational point of X_k.

Let us fix, for the rest of this chapter, a set $\tilde{S}$ of closed points of X_K in bijection with the set X_k^{cl} of closed points of X_k, defined as in Definition 3.3.7. Suppose we are given a section $s : G_K \to \pi_1(X_K - \tilde{S})$, where $\pi_1(X_K - \tilde{S})$ is defined as in Definition 3.3.1. There exists a surjective specialisation homomorphism $\mathrm{Sp} : \pi_1(X_K - \tilde{S}) \twoheadrightarrow G_{X_k}$ (Theorem 3.3.8), and we denote $\varphi_s := \mathrm{Sp} \circ s$.

$$\begin{tikzcd}
\pi_1(X_K - \tilde{S}) \arrow[r] \arrow[d, two heads, "\mathrm{Sp}"'] & G_K \arrow[l, bend right, "s"'] \arrow[d, two heads, "\rho"] \arrow[dl, bend right, "\varphi_s"'] \\
G_{X_k} \arrow[r] & G_k
\end{tikzcd} \tag{3.15}$$

Definition 3.4.13. We say the section $s : G_K \to \pi_1(X_K - \tilde{S})$ is *unramified* if $\varphi_s(I_K) = 0$. Otherwise we say s is *ramified.*

An unramified section $s : G_K \to \pi_1(X_K - \tilde{S})$ naturally induces a section $\bar{s} : G_k \to G_{X_k}$ defined by $\bar{s}(\sigma I_K) = \varphi_s(\sigma)$ for every $\sigma \in G_K$, where σI_K is the coset of σ modulo I_K. This induced section is called the *specialisation of s.*

For an unramified section s of $\pi_1(X_K - \tilde{S})$, we will denote its specialisation by $\bar{s}$.

Recall from Definition 3.3.1 that

$$\pi_1(X_K - \tilde{S}) = \varprojlim_{B \subset \tilde{S} \text{ finite}} \pi_1(X_K - B, \bar{\xi}_1)$$

where $\bar{\xi}_1$ is a geometric point with image the generic point of X_K, and each B is a finite subset of $\tilde{S}$. For some finite subset $B \subset \tilde{S}$, let S denote the closure of B in X, which is a divisor on X that is finite étale over R such that $S_K = B$ (see the paragraph before Theorem 3.3.8). Denote $U_K := X_K - S_K$ and $U_k := X_k - S_k$. Then a section $s : G_K \to \pi_1(X_K - \tilde{S})$ induces a section $s_U : G_K \to \pi_1(U_K, \bar{\xi}_1)$.

For $B' \subset \tilde{S}$ another finite subset with $B \subset B'$, write similarly S' for the closure of B' in X, and denote $U'_K = X_K - S'_K$ and $U'_k = X_k - S'_k$. Then the section s also induces a section $s_{U'} : G_K \to \pi_1(U'_K, \bar{\xi}_1)$, which, moreover, is compatible with s_U under the homomorphism $\mathrm{pr}_{U_K} : \pi_1(U'_K, \bar{\xi}_1) \twoheadrightarrow \pi_1(U_K, \bar{\xi}_1)$, which is to say that, in the following diagram, $\mathrm{pr}_{U_K} \circ s_{U'} = s_U$.

$$\begin{tikzcd}
 & & \pi_1(X_K - \tilde{S}) \arrow[dl, two heads] \arrow[d, two heads] \arrow[rr] & & G_K \arrow[ll, bend right, "s"'] \arrow[d, equal] \\
\cdots \arrow[r, two heads] & \pi_1(U'_K, \bar{\xi}_1) \arrow[r, two heads, "\mathrm{pr}_{U_K}"'] & \pi_1(U_K, \bar{\xi}_1) \arrow[r, two heads] & \cdots \arrow[r] & G_K \arrow[lll, bend right, "s_{U'}"'] \arrow[ll, bend right, "s_U"']
\end{tikzcd} \tag{3.16}$$

Thus the section $s : G_K \to \pi_1(X_K - \tilde{S})$ defines a system of sections $\{s_U\}$ of the quotients $\pi_1(U_K, \bar{\xi}_1)$ of $\pi_1(X_K - \tilde{S})$ which are compatible with the homomorphisms $\mathrm{pr}_{U_K} : \pi_1(U'_K, \bar{\xi}_1) \twoheadrightarrow \pi_1(U_K, \bar{\xi}_1)$, for finite subsets $B \subset B'$ of $\tilde{S}$. Conversely, such a compatible system defines a section of $\pi_1(X_K - \tilde{S})$.

For U_K and U'_K as above, there exist surjective specialisation homomorphisms $\mathrm{Sp}_U : \pi_1(U_K, \bar{\xi}_1) \twoheadrightarrow \pi_1(U_k, \bar{\xi}_2)$ and $\mathrm{Sp}_{U'} : \pi_1(U'_K, \bar{\xi}_1) \twoheadrightarrow \pi_1(U'_k, \bar{\xi}_2)$, where $\bar{\xi}_2$ is a geometric point with image the generic point of X_k. The homomorphisms $\varphi_U := \mathrm{Sp}_U \circ s_U$ and $\varphi_{U'} := \mathrm{Sp}_{U'} \circ s_{U'}$ are compatible with the homomorphism $\mathrm{pr}_{U_k} : \pi_1(U'_k, \bar{\xi}_2) \twoheadrightarrow$

$\pi_1(U_k, \bar{\xi}_2)$, meaning that, in the following diagram, $\mathrm{pr}_{U_k} \circ \varphi_{U'} = \varphi_U$.

$$\begin{tikzcd}
& & \pi_1(X_K - \tilde{S}) \arrow[rr] \arrow[dl, two heads] \arrow[d, two heads] & & G_K \arrow[ll, bend right, "s"'] \arrow[d, equal] \\
\cdots \arrow[r, two heads] & \pi_1(U'_K, \bar{\xi}_1) \arrow[r, two heads, "\mathrm{pr}_{U_K}"] \arrow[d, two heads, "\mathrm{Sp}_{U'}"'] & \pi_1(U_K, \bar{\xi}_1) \arrow[r, two heads] \arrow[d, two heads, "\mathrm{Sp}_U"'] & \cdots \arrow[r] & G_K \arrow[lll, bend right, "s_{U'}"'] \arrow[ll, bend right, "s_U"'] \arrow[dlll, bend right, "\varphi_{U'}"'] \arrow[dll, bend right, "\varphi_U"'] \arrow[d, two heads, "\rho"] \\
\cdots \arrow[r, two heads] & \pi_1(U'_k, \bar{\xi}_2) \arrow[r, two heads, "\mathrm{pr}_{U_k}"] & \pi_1(U_k, \bar{\xi}_2) \arrow[r, two heads] & \cdots \arrow[r] & G_k
\end{tikzcd} \tag{3.17}$$

Thus the section $s : G_K \to \pi_1(X_K - \tilde{S})$ defines a system of homomorphisms $\{\varphi_U\}$ which are compatible with the homomorphisms $\mathrm{pr}_{U_k} : \pi_1(U'_k, \bar{\xi}_2) \twoheadrightarrow \pi_1(U_k, \bar{\xi}_2)$. In particular, the inverse limit $\varprojlim_{B \subset \tilde{S} \text{ finite}} \varphi_U$ is exactly the homomorphism φ_s of diagram 3.15.

Lemma 3.4.14. *A section $s : G_K \to \pi_1(X_K - \tilde{S})$ is ramified if and only if there is an open subset $U_k \subset X_k$ for which the section $s_U : G_K \to \pi_1(U_K, \bar{\xi}_1)$ induced by s is ramified.*

Proof. Since $\varphi_s(I_K) = \varprojlim_{U_k \subset X_k \text{ open}} \varphi_U(I_K)$, $\varphi_s(I_K)$ is trivial if and only if $\varphi_U(I_K)$ is trivial for every open subset $U_k \subset X_k$. □

Proposition 3.4.15. *Assume that k satisfies condition (ii) of Definition 2.3.1. Let $U'_k \subset U_k \subset X_k$ be any two open subsets, and let $\tilde{X}_{U_k}$, resp. $\tilde{X}_{U'_k}$ be the normalisation of X_k in some universal pro-étale cover $\tilde{U}_k \to U_k$, resp. $\tilde{U}'_k \to U'_k$.*

Suppose that the section s_U is ramified, with $\varphi_U(I_K)$ contained in the inertia subgroup $I_{\tilde{x}_U} \subset \pi_1(U_k, \bar{\xi}_2)$ for some $x \in S(k)$ and some $\tilde{x}_U$ in $\tilde{X}_{U_k}$ above x (see Lemma 3.4.11). Then the section $s_{U'}$ is ramified, and $\varphi_{U'}(I_K)$ is contained in the inertia subgroup $I_{\tilde{x}_{U'}} \subset \pi_1(U'_k, \bar{\xi}_2)$ of some $\tilde{x}_{U'}$ in $\tilde{X}_{U'_k}$ above the same point $x \in S'(k)$.

Proof. If $U'_k \subset U_k$ then $S'_k \supset S_k$, so S'_k contains x. If $\varphi_{U'}(I_K)$ is trivial, then its image under the homomorphism $\mathrm{pr}_{U_k} : \pi_1(U'_k, \bar{\xi}_2) \twoheadrightarrow \pi_1(U_k, \bar{\xi}_2)$ must also be trivial, but by compatibility of the homomorphisms φ_U (see above discussion), this image coincides with $\varphi_U(I_K)$, which is nontrivial by assumption.

Thus $\varphi_{U'}(I_K)$ must be non-trivial, so, by Lemma 3.4.11, it is contained in an inertia subgroup $I_{\tilde{z}_{U'}}$ for some $z \in S'(k)$ and some $\tilde{z}_{U'}$ in $\tilde{X}_{U'_k}$ above z. Suppose $z \neq x$. By Lemma 1.4.18, if $z \in S_k$ then the image of $I_{\tilde{z}_{U'}}$ under pr_{U_k} is an inertia subgroup $I_{\tilde{z}_U} \subset \pi_1(U_k, \bar{\xi}_2)$ for some $\tilde{z}_U$ in $\tilde{X}_{U_k}$ above z, which intersects trivially with $I_{\tilde{x}_U}$ by Proposition 1.4.17 (i). Meanwhile, if $z \notin S_k$ the image of $I_{\tilde{z}_{U'}}$ in $\pi_1(U_k, \bar{\xi}_2)$ is trivial by Proposition 1.4.16. Both of these contradict compatibility of φ_U and $\varphi_{U'}$, so we must have $z = x$. □

Intuitively, when we pass from U_k to U'_k we add points to S_k, hence we add inertia groups into $\pi_1(U_k, \bar{\xi}_2)$. But in doing this we would not expect $\varphi_U(I_K)$ to become trivial or contained in a different inertia subgroup. The above Proposition verifies this intuition in the case where k satisfies condition (ii) of Definition 2.3.1.

The ramification of a section s of $\pi_1(X_K - \tilde{S})$ is therefore characterised by the ramification of the system of sections s_U it induces. There are two cases - either all the s_U in the system are unramified, in which case s is unramified; or at some U_k the section s_U becomes ramified associated to some closed point $x \in X_k$, and then it stays ramified as we enlarge S_k, associated to the same point x if k satisfies condition (ii) of Definition 2.3.1.

Let us fix a universal pro-étale cover $\tilde{X}_{\tilde{S}} \to X_K - \tilde{S}$ (see Definition 3.3.2), and denote by $k(X_k)^{\mathrm{sep}}$ the separable closure of the function field $k(X_k)$ determined by the geometric point $\bar{\xi}_2$.

Lemma 3.4.16. *Assume that k satisfies condition (ii) of Definition 2.3.1. If s is ramified then $\varphi_s(G_K) \subset D_{\tilde{x}}$ for a unique valuation $\tilde{x}$ on $k(X_k)^{\mathrm{sep}}$ extending a k-rational point x of X_k, and $s(G_K) \subset D_{\tilde{y}}$ for some $\tilde{y}$ in $\tilde{X}_{\tilde{S}}$ above the unique K-rational point y in $\tilde{S}$ specialising to x.*

See Definition 3.3.3 and Remark 3.3.4 for the definition of the decomposition subgroups of $\pi_1(X_K - \tilde{S})$. See also Definition 1.4.22 for the definition of the decomposition subgroups of G_{X_k} and the meaning of an extension of a k-rational point of X_k to $k(X_k)^{\mathrm{sep}}$.

Proof. Lemma 3.4.14 and Proposition 3.4.15 imply that, for some open subset $U_k \subset X_k$, $s_{U'}$ is ramified for every open subset $U'_k \subset X_k$ contained in U_k. For each such U'_k, let S'_k denote the complement $X_k - U'_k$, $S'_K \subset \tilde{S}$ the finite subset consisting of those points of $\tilde{S}$ specialising to S'_k, and $U'_K := X_K - S'_K$. Choose universal pro-étale covers $\tilde{U}'_k \to U'_k$ and $\tilde{U}'_K \to U'_K$, and denote by $\tilde{X}_{U'_k}$ and $\tilde{X}_{U'_K}$ the normalisations of X_k and X_K in $\tilde{U}'_k$ and $\tilde{U}'_K$ respectively.

By Proposition 3.4.8, for each such U'_k we have $\varphi_{U'}(I_K) \subset I_{\tilde{x}_{U'}}$ for a unique $\tilde{x}_{U'}$ in $\tilde{X}_{U'_k}$ above a k-rational point x of S'_k. As in the proof of Lemma 3.4.11, this implies that $\varphi_{U'}(G_K)$ is contained in the decomposition subgroup $D_{\tilde{x}_{U'}}$, and $s_{U'}(G_K) \subset D_{\tilde{y}_{U'}}$ for some $\tilde{y}_{U'}$ in $\tilde{X}_{U'_K}$ above the unique K-rational point y of S'_K specialising to x. By Proposition 3.4.15, the points $x \in X(k)$, $y \in X(K)$ are the same for every such U'_k, thus taking the inverse limit over the open subsets of X_k gives the result of the Lemma. The uniqueness of $\tilde{x}$ follows from [NSW08, Corollary 12.1.3] (see also Remark 2.2.3). □

Lemma 3.4.17. *Suppose s is unramified and its specialisation $\bar{s}$ is geometric with $\bar{s}(G_k) \subset D_{\tilde{x}}$ for some $x \in X(k)$ and some extension $\tilde{x}$ of x to $k(X_k)^{\mathrm{sep}}$. Then $s(G_K) \subset D_{\tilde{y}}$ for some $\tilde{y}$ in $\tilde{X}_{\tilde{S}}$ above the unique (K-rational) point y of $\tilde{S}$ specialising to x.*

Proof. Let $B \subset \tilde{S}$ be a finite subset of $\tilde{S}$ with closure S in X such that $x \in S_k$. Denoting $U = X - S$, the section s induces a section $s_U : G_K \to \pi_1(U_K, \bar{\xi}_1)$, which is unramified by Lemma 3.4.14. Let $\tilde{X}_{U_k}$, respectively $\tilde{X}_{U_k}$ denote the normalisation of X_K, resp. X_k in some universal pro-étale cover of U_K, resp. U_k. By commutativity of diagram 3.17, $\bar{s}_U(G_k) \subset D_{\tilde{x}_U}$ for some $\tilde{x}_U$ in $\tilde{X}_{U_k}$ above x. Hence, by Lemma 3.4.10, $s_U(G_K) \subset D_{\tilde{y}_U}$ for some $\tilde{y}$ in $\tilde{X}_{U_K}$ above the unique point y of S_K specialising to x. This is true for every finite subset of $\tilde{S}$, so taking the inverse limit gives the statement of the Lemma. □

A section $s : G_K \to \pi_1(X_K - \tilde{S})$ induces, in particular, a section $s_X : G_K \to \pi_1(X_K, \bar{\xi}_1)$ of the étale fundamental group of the projective curve X_K. The section s_X is always unramified, by Proposition 3.4.8 (see also Remark 3.4.9), thus it induces a section $\bar{s}_X : G_k \to \pi_1(X_k, \bar{\xi}_2)$.

$$\begin{tikzcd}
\pi_1(X_K - \tilde{S}) \arrow[r] \arrow[d, two heads] & G_K \arrow[l, bend right, "s"'] \arrow[d, equal] \\
\pi_1(X_K, \bar{\xi}_1) \arrow[r] \arrow[d, two heads, "\mathrm{Sp}_X"'] & G_K \arrow[l, bend right, "s_X"'] \arrow[d, two heads, "\rho"] \\
\pi_1(X_k, \bar{\xi}_2) \arrow[r] & G_k \arrow[l, bend right, "\bar{s}_X"']
\end{tikzcd}$$

Proposition 3.4.18. *Assume that k satisfies conditions (ii) and (iii) (a) of Definition 2.3.1, and assume the birational section conjecture holds for X_k. Let $s : G_K \to \pi_1(X_K - \tilde{S})$ be a section, $s_X : G_K \to \pi_1(X_K, \bar{\xi}_1)$ the unramified section of $\pi_1(X_K, \bar{\xi}_1)$ induced as above by s, and $\bar{s}_X$ the specialisation of s_X. Let $\tilde{X}_K \to X_K$ and $\tilde{X}_k \to X_k$ be universal pro-étale covers. Then $\bar{s}_X(G_k) \subset D_{\tilde{x}}$ for a unique k-rational point x of X_k and some $\tilde{x}$ in $\tilde{X}_k$ above x, and $s_X(G_K) \subset D_{\tilde{y}}$ for some $\tilde{y}$ in $\tilde{X}_K$ above the unique (K-rational) point y of $\tilde{S}$ specialising to x.*

Proof. We have the following commutative diagram.

$$\begin{tikzcd}
\pi_1(X_K - \tilde{S}) \arrow[r, two heads] \arrow[d, two heads, "\mathrm{Sp}_U"'] & \pi_1(X_K, \bar{\xi}_1) \arrow[r] \arrow[d, two heads, "\mathrm{Sp}_X"'] & G_K \arrow[l, bend right, "s_X"'] \arrow[ll, bend right, "s"'] \arrow[d, two heads, "\rho"] \arrow[dll, bend left, "\varphi_s"'] \arrow[dl, bend left, "\varphi_X"'] \\
G_{X_k} \arrow[r, two heads] & \pi_1(X_k, \bar{\xi}_2) \arrow[r] & G_k \arrow[l, bend right, "\bar{s}_X"']
\end{tikzcd}$$

If s is ramified then, by Lemma 3.4.16, $\varphi_s(G_K) \subset D_{\tilde{x}'}$ for a unique k-rational point x of X_k and a (unique) extension $\tilde{x}'$ of x to $k(X_k)^{\mathrm{sep}}$, and $s(G_K) \subset D_{\tilde{y}'}$ some $\tilde{y}'$ in $\tilde{X}_{\tilde{S}}$ above the unique K-rational point y of $\tilde{S}$ specialising to x. By commutativity of the above diagram, therefore, $s_X(G_K) = D_{\tilde{y}}$ for some $\tilde{y}$ in $\tilde{X}_K$ above y, and $\bar{s}_X(G_k) = D_{\tilde{x}}$ for some $\tilde{x}$ in $\tilde{X}_k$ above x (see also diagrams 3.16 and 3.17 and their surrounding paragraphs).

If s is unramified then, by the assumption that the birational section conjecture holds for X_k, the specialisation $\bar{s} : G_k \to G_{X_k}$ satisfies $\bar{s}(G_k) \subset D_{\tilde{x}'}$ for a unique

$x \in X(k)$ and a (unique) extension $\tilde{x}'$ of x to $k(X_k)^{\mathrm{sep}}$. Then, by Lemma 3.4.17, $s(G_K) \subset D_{\tilde{y}'}$ for some $\tilde{y}'$ in $\tilde{X}_{\tilde{S}}$ above the unique point y of $\tilde{S}$ specialising to x. By commutativity of the above diagram (or of diagram 3.17), this implies $s_X(G_K) = D_{\tilde{y}}$ for some $\tilde{y}$ in $\tilde{X}_K$ above y, and $\bar{s}_X(G_k) = D_{\tilde{x}}$ for some $\tilde{x}$ in $\tilde{X}_k$ above x.

It remains to prove that x is the unique k-rational point of X_k such that $\bar{s}_X(G_k) \subset D_{\tilde{x}}$ for some $\tilde{x}$ in $\tilde{X}_k$ above x. But this follows immediately from Proposition 1.6.8 and condition (iii) (a) of Definition 2.3.1 (see the proof of Lemma 3.4.11). □

Chapter 4

Sections over function fields

Let k be a field and let C a smooth, separated, connected curve over k with function field K. Let $\mathcal{X} \to C$ be a smooth relative curve, with generic fibre $X := \mathcal{X} \times_C \operatorname{Spec} K$ which is a geometrically connected curve over K. For each closed point c of C, denote by $\mathcal{X}_c := \mathcal{X} \times_C \operatorname{Spec} k(c)$ the closed fibre of $\mathcal{X}$ at c. We have a diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & \operatorname{Spec} K \arrow[d] \\
\mathcal{X} \arrow[r] & C \\
\mathcal{X}_c \arrow[r] \arrow[u] & \operatorname{Spec} k(c) \arrow[u]
\end{tikzcd}$$

In this chapter we explore under what circumstances sections of the absolute Galois group G_X of X induce sections of the absolute Galois group $G_{\mathcal{X}_c}$ of the closed fibre at c. To do this we pass to the local situation by taking the completion of K with respect to the valuation corresponding to the point $c \in C^{\mathrm{cl}}$, and we may then use the results from chapter 3.

We also investigate the étale abelian parts of such sections, which enables us to get back to the global setting under the condition that the Tate module of the Shafarevich-Tate group $\text{Ш}(\mathcal{J})$ of the relative Jacobian of $\mathcal{X}$ is trivial.

4.1 Specialisation of sections

We start with some general definitions. Let Y be a smooth, projective, geometrically connected curve over a field F. For any valuation ν on F, let F_ν denote the completion of F with respect to ν, and let Y_ν denote the base change $Y \times_{\operatorname{Spec} F} \operatorname{Spec} F_\nu$. Let $\overline{F_\nu}$ be an algebraic closure of F_ν, and denote by $\overline{F}$ the algebraic closure of F in $\overline{F_\nu}$. Set-theoretically, an element $y : \operatorname{Spec} \overline{F} \to Y$ of $Y(\overline{F})$ maps the unique point of $\operatorname{Spec} \overline{F}$ to a closed point of Y, which we will refer to the "image of y in Y^{cl}". Similarly, we may speak of the image of an element of $Y_\nu(\overline{F_\nu})$ in Y_ν^{cl}.

By the universal property of pullbacks, an element of $Y(\overline{F})$ uniquely defines and element of $Y_\nu(\overline{F_\nu})$, hence there is an inclusion $i : Y(\overline{F}) \hookrightarrow Y_\nu(\overline{F_\nu})$.

Definition 4.1.1. With the above notation, an *algebraic point* of Y_ν is the image in Y_ν^{cl} of an element of $i(Y(\overline{F}))$. A *transcendental point* of Y_ν is the image in Y_ν^{cl} of an element of $Y_\nu(\overline{F_\nu}) \backslash i(Y(\overline{F}))$.

Let $V \subset Y$ be an open subset of Y, with complement $T = \{y_1, \dots y_n\}$ a finite set of closed points of Y. Each $y_i \in T$ is the image in Y^{cl} of an element of $Y(\overline{F})$, which we denote $y_i : \operatorname{Spec} \overline{F} \to Y$ (such an element is not unique). By the pullback property, this naturally determines an element of $Y_\nu(\overline{F_\nu})$, which has image a closed point $y_{\nu,i}$ of Y_ν that maps to y_i under the projection $Y_\nu \to Y$. By the above definition, the closed point $y_{\nu,i}$ is an algebraic point of Y_ν. Thus the base change $T_\nu := T \times_{\operatorname{Spec} F} \operatorname{Spec} F_\nu$ consists of algebraic points, and $V_\nu = V \times_{\operatorname{Spec} F} \operatorname{Spec} F_\nu = Y_\nu - T_\nu$ contains all the transcendental points of Y_ν.

Let us denote by Y_ν^{tr} the complement in Y_ν of the set of all algebraic points of Y_ν. As we enlarge the closed set $T \subset Y$, the open subset $V_\nu \subset Y_\nu$ approaches Y_ν^{tr}.

Definition 4.1.2. With the above notation, let $\bar{\xi}_\nu : \operatorname{Spec} \overline{K(Y_\nu)} \to Y_\nu$ be a geometric point with image the generic point of Y_ν. Define the group $\pi_1(Y_\nu^{\mathrm{tr}})$ to be the inverse limit

$$\pi_1(Y_\nu^{\mathrm{tr}}) := \varprojlim_{V \subset Y \text{ open}} \pi_1(V_\nu, \bar{\xi}_\nu)$$

where the limit is taken over all open subsets $V \subset Y$, ordered by inclusion, and V_ν denotes the base change $V \times_{\operatorname{Spec} F} \operatorname{Spec} F_\nu$.

Remark 4.1.3. While we omit the base point $\bar{\xi}_\nu$ in the notation $\pi_1(Y_\nu^{\mathrm{tr}})$, it is implicit that this group comes with this choice of base point (compare with Remark 3.3.4). Thus $\pi_1(Y_\nu^{\mathrm{tr}})$ is naturally a quotient of G_{Y_ν} by Lemma 1.4.21.

Now we return to the notation of the beginning of this chapter. Let k be a field and let C a smooth, separated, connected curve over k with function field K. Let $\mathcal{X} \to C$ be a smooth relative curve, with generic fibre $X := \mathcal{X} \times_C \operatorname{Spec} K$ which is a geometrically connected curve over K. For some $c \in C^{\mathrm{cl}}$, let K_c denote the completion of K with respect to the valuation of K corresponding to the closed point $c \in C^{\mathrm{cl}}$. Denote by $X_c := X \times_{\operatorname{Spec} K} \operatorname{Spec} K_c$ the base change of X to K_c. Let $\bar{\xi}_c : \operatorname{Spec} \Omega_c \to X_c$ be a geometric point with image the generic point of X_c. Then $\bar{\xi}_c$ maps to a geometric point $\bar{\xi} : \operatorname{Spec} \Omega \to X$ of X with image the generic point of X. Denoting by $\overline{K_c}$, respectively $\overline{K}$ the algebraic closure of K_c in Ω_c, resp. of $\overline{K}$ in Ω, we denote the induced geometric points of $X_{\overline{K_c}}$ and $X_{\overline{K}}$ again by $\bar{\xi}_c$ and $\bar{\xi}$, respectively.

Let $U \subset X$ be an open subset of X, with complement $S = \{x_1, ..., x_n\}$ a finite set of closed points of X. Then $S_c := S \times_{\operatorname{Spec} K} \operatorname{Spec} K_c$ consists of algebraic points of X_c, and $U_c := U \times_{\operatorname{Spec} K} \operatorname{Spec} K_c = X_c - S_c$ contains all the transcendental points of X_c. Similarly, the points of $S_{\overline{K_c}}$ are algebraic points of $Y_{\overline{K_c}}$, and $U_{\overline{K_c}} = Y_{\overline{K_c}} - S_{\overline{K_c}}$ contains all the transcendental points of $Y_{\overline{K_c}}$.

Functoriality of the fundamental group yields a diagram

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(U_{\overline{K_c}}, \bar{\xi}_c) \arrow[r] \arrow[d] & \pi_1(U_c, \bar{\xi}_c) \arrow[r] \arrow[d] & G_{K_c} \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \pi_1(U_{\overline{K}}, \bar{\xi}) \arrow[r] & \pi_1(U, \bar{\xi}) \arrow[r] & G_K \arrow[r] & 1
\end{tikzcd} \tag{4.1}$$

where the rows are the fundamental exact sequences.

Lemma 4.1.4. *If K has characteristic zero, the left vertical map in diagram 4.1 is an isomorphism, and the right square is a pullback square.*

Proof. For a proof of the first statement we refer to [Sza09], "Second proof of Corollary 5.7.6", as well as Remark 5.7.8 after it. The second statement follows from the first as in the proof of Lemma 3.4.2. □

A section $s_U : G_K \to \pi_1(U, \bar{\xi})$ therefore pulls back to a section $s_{U_c} : G_{K_c} \to \pi_1(U_c, \bar{\xi}_c)$ (see the paragraph after Corollary 3.4.5).

Taking the projective limit of the homomorphisms $\pi_1(U_c, \bar{\xi}_c) \to \pi_1(U, \bar{\xi})$ over the open subsets of X yields a homomorphism $\pi_1(X_c^{\mathrm{tr}}) \to G_X$, and similarly, taking the projective limit of the homomorphisms $\pi_1(U_{\overline{K_c}}, \bar{\xi}_c) \to \pi_1(U_{\overline{K}}, \bar{\xi})$ over the open subsets of $X_{\overline{K}}$ yields a homomorphism $\pi_1(X^{\mathrm{tr}}_{\overline{K_c}}) \to G_{X_{\overline{K}}}$ (see Lemma 1.4.21 and Definition 4.1.2). Thus we obtain the following commutative diagram of exact sequences.

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(X^{\mathrm{tr}}_{\overline{K_c}}) \arrow[r] \arrow[d] & \pi_1(X^{\mathrm{tr}}_c) \arrow[r] \arrow[d] & G_{K_c} \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & G_{X_{\overline{K}}} \arrow[r] & G_X \arrow[r] & G_K \arrow[r] & 1
\end{tikzcd} \tag{4.2}$$

Lemma 4.1.5. *If K has characteristic zero, the left vertical map in diagram 4.2 is an isomorphism, and the right square is a pullback square.*

Proof. Follows from Lemma 4.1.4 on taking the projective limit over the open subsets of X. □

Thus, when K has characteristic zero, a section $s : G_K \to G_X$ pulls back to a section $s_c : G_{K_c} \to \pi_1(X^{\mathrm{tr}}_c)$.

Denote by $\mathcal{X}_c := \mathcal{X} \times_C \operatorname{Spec} k(c)$ the closed fibre of $\mathcal{X}$ at c. Let $\mathcal{O}_c$ denote the valuation ring of K_c, and write $\hat{\mathcal{X}}_c := \mathcal{X} \times_C \operatorname{Spec} \mathcal{O}_c$. Thus $\hat{\mathcal{X}}_c$ has closed fibre $\mathcal{X}_c$ and generic fibre X_c, as illustrated by the following diagram.

$$\begin{tikzcd}
X_c \arrow[r] \arrow[d] & \operatorname{Spec} K_c \arrow[d] \\
\hat{\mathcal{X}}_c \arrow[r] & \operatorname{Spec} \mathcal{O}_c \\
\mathcal{X}_c \arrow[r] \arrow[u] & \operatorname{Spec} k(c) \arrow[u]
\end{tikzcd}$$

The following Lemma and Definition are directly comparable to Lemma 3.3.5 and Definition 3.3.7, the difference being that we are now concerned with algebraic points of X_c.

Lemma 4.1.6. *For each closed point x of $\mathcal{X}_c$, there exists an algebraic point y of X_c specialising to x whose residue field is the unique unramified extension L_c of K_c whose valuation ring $\mathcal{O}_{L_c}$ has residue field $k(x)$.*

Proof. Recall that Lemma 3.3.5 implies existence of an L_c-rational point of X_c specialising to x. To prove that we can choose such a point to be algebraic, we now take a different approach.

Let π_c denote the uniformiser of $\mathcal{O}_c$, π_{L_c} the image under the inclusion $\mathcal{O}_c \hookrightarrow \mathcal{O}_{L_c}$ (which is a uniformiser of $\mathcal{O}_{L_c}$ since $\mathcal{O}_{L_c}$ is unramified over $\mathcal{O}_c$), and $\mathfrak{m}_{L_c}$ the maximal ideal $(\pi_{L_c}) \subset \mathcal{O}_{L_c}$. For ease of notation, let us write $\hat{\mathcal{X}}'_c := \hat{\mathcal{X}}_c \times_{\operatorname{Spec} \mathcal{O}_c} \operatorname{Spec} \mathcal{O}_{L_c}$, and similarly $X'_c := X_c \times_{\operatorname{Spec} K_c} \operatorname{Spec} L_c$ and $\mathcal{X}'_c := \mathcal{X}_c \times_{\operatorname{Spec} k(c)} \operatorname{Spec} k(x)$. This is illustrated by the following diagram.

$$\begin{tikzcd}
X'_c \arrow[r] \arrow[d] & \operatorname{Spec} L_c \arrow[d] \\
\hat{\mathcal{X}}'_c \arrow[r] & \operatorname{Spec} \mathcal{O}_{L_c} \\
\mathcal{X}'_c \arrow[r] \arrow[u] & \operatorname{Spec} k(x) \arrow[u]
\end{tikzcd}$$

Let x' be a point of $\mathcal{X}'_c$ which maps to x under the projection $\mathcal{X}'_c \to \mathcal{X}_c$. A point $y' \in X'_c$ specialising to x' corresponds to a prime ideal of $\mathcal{O}_{\hat{\mathcal{X}}'_c,x'}$ of height 1 which does not contain π_{L_c}. This remains true when we pass to the $\mathfrak{m}_{x'}$-adic completion $\hat{\mathcal{O}}_{\hat{\mathcal{X}}'_c,x'}$, where $\mathfrak{m}_{x'}$ is the maximal ideal of $\mathcal{O}_{\hat{\mathcal{X}}'_c,x'}$. This is because $\mathfrak{m}_{x'}\hat{\mathcal{O}}_{\hat{\mathcal{X}}'_c,x'}$ is the maximal ideal of $\hat{\mathcal{O}}_{\hat{\mathcal{X}}'_c,x'}$, and $\dim \mathcal{O}_{\hat{\mathcal{X}}'_c,x'} = \dim \hat{\mathcal{O}}_{\hat{\mathcal{X}}'_c,x'}$.

We have an isomorphism $\hat{\mathcal{O}}_{\hat{\mathcal{X}}'_c,x'} \simeq \mathcal{O}_{L_c}[[T]]$ for some parameter T, under which the maximal ideal of $\hat{\mathcal{O}}_{\hat{\mathcal{X}}'_c,x'}$ corresponds to $\mathfrak{m}_{L_c} + (T) = (\pi_{L_c}, T)$. The ring $\mathcal{O}_{L_c}[[T]]$ is a local ring of dimension 2, and it is a regular local UFD since $\mathcal{O}_{L_c}$ is. Thus all prime ideals of $\mathcal{O}_{L_c}[[T]]$ which are not maximal or the zero ideal have height 1, and since it is a UFD, every such prime ideal is generated by an irreducible element. By the Weierstrass Preparation Theorem, any element $f \in \mathcal{O}_{L_c}[[T]]$ can be written uniquely in the form uF, where u is a unit in $\mathcal{O}_{L_c}[[T]]$ and F is a monic polynomial whose coefficients are all in $\mathfrak{m}_{L_c}$. Therefore, f is an irreducible element of $\mathcal{O}_{L_c}[[T]]$ if and only if F is an irreducible polynomial in $\mathcal{O}_{L_c}[T]$.

To choose an L_c-rational point y' of X'_c which specialises to x' is therefore to choose a prime ideal of $\mathcal{O}_{L_c}[[T]]$ of the form $(T - \alpha)$, where $\alpha \in \mathfrak{m}_{L_c}$. That is, the set of L_c-rational points of X'_c specialising to x' is in bijection with $\mathfrak{m}_{L_c}$. Let L be a finite extension of K whose completion is L_c and which is unramified with respect to the valuation corresponding to c. Then choosing α to be an element of $\mathfrak{m}_{L_c} \cap L$ yields an L_c-rational algebraic point y' of X'_c which specialises to x'. The image y of

y' under the projection $X_c' \to X_c$ is then an L_c-rational algebraic point of X_c which specialises to x. $\square$

Definition 4.1.7. For each closed point x of $\mathcal{X}_c$, fix a choice of algebraic point $y \in X_c^{\mathrm{cl}}$ which specialises to x and whose residue field is the unique unramified extension of K whose valuation ring has residue field $k(x)$ (such a point exists by Lemma 4.1.6 above). We define $\tilde{S}$ to be the set of these chosen algebraic points $y \in X_c^{\mathrm{cl}}$. Thus, $\tilde{S}$ is a subset of X_c^{cl} which consists of algebraic points and which is in bijection with $\mathcal{X}_c^{\mathrm{cl}}$.

We denote by $\tilde{S}_{\overline{K_c}}$ the base change of $\tilde{S}$ to $\overline{K_c}$. Thus $\tilde{S}_{\overline{K_c}}$ is a subset of $X_{\overline{K_c}}^{\mathrm{cl}}$ in bijection with $\mathcal{X}_{\overline{k(c)}}$. The points of $\tilde{S}_{\overline{K_c}}$ are algebraic points, since they are defined by sections in the image of the composite map $X(\overline{K}) \hookrightarrow X_c(\overline{K_c}) \simeq X_{\overline{K_c}}(\overline{K_c})$.

By Lemma 3.2.1, for every finite subset $B \subset \tilde{S}$, the closure of B in $\hat{\mathcal{X}}_c$ is a divisor which is finite étale over $\mathcal{O}_c$. Thus, under the condition that k has characteristic zero, we deduce the existence of specialisation homomorphisms as in the following Theorem.

Theorem 4.1.8. *Assume that k has characteristic zero. With $\tilde{S}$ and $\tilde{S}_{\overline{K_c}}$ as in Definition 4.1.7, there exist surjective homomorphisms* Sp *and ρ and an isomorphism* $\overline{\mathrm{Sp}}$ *making the following diagram commutative.*

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(X_{\overline{K_c}} - \tilde{S}_{\overline{K_c}}) \arrow[r] \arrow[d, "\overline{\mathrm{Sp}}"', "\wr"] & \pi_1(X_c - \tilde{S}) \arrow[r] \arrow[d, two heads, "\mathrm{Sp}"'] & G_K \arrow[r] \arrow[d, two heads, "\rho"'] & 1 \\
1 \arrow[r] & G_{\mathcal{X}_{\overline{k(c)}}} \arrow[r] & G_{\mathcal{X}_c} \arrow[r] & G_{k(c)} \arrow[r] & 1
\end{tikzcd}$$

Proof. This is just a re-statement of Theorem 3.3.8. $\square$

Since we have chosen $\tilde{S}$ to consist of algebraic points, $\pi_1(X_c - \tilde{S})$ is naturally a quotient of $\pi_1(X_c^{\mathrm{tr}})$. Hence, under the conditions of Theorem 4.1.8, we have the following commutative diagram.

$$
\begin{tikzcd}
1 \arrow[r] & \pi_1(X^{\mathrm{tr}}_{\overline{K_c}}) \arrow[r] \arrow[d, two heads] & \pi_1(X^{\mathrm{tr}}_c) \arrow[r] \arrow[d, two heads] & G_{K_c} \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \pi_1(X_{\overline{K_c}} - \tilde{S}_{\overline{K_c}}) \arrow[r] \arrow[d, "\overline{\mathrm{Sp}}"', "\wr"] & \pi_1(X_c - \tilde{S}) \arrow[r] \arrow[d, two heads, "\mathrm{Sp}"'] & G_{K_c} \arrow[r] \arrow[d, two heads, "\rho"'] & 1 \\
1 \arrow[r] & G_{\mathcal{X}_{\overline{k(c)}}} \arrow[r] & G_{\mathcal{X}_c} \arrow[r] & G_{k(c)} \arrow[r] & 1
\end{tikzcd}
$$

We will assume for the rest of this section that X is hyperbolic and that k has characteristic zero. Note that since $\mathcal{X} \to C$ is flat, hyperbolicity of X implies that every closed fibre $\mathcal{X}_c$ is hyperbolic. Let $s : G_K \to G_X$ be a section, and let $s_c : G_{K_c} \to \pi_1(X_c^{\mathrm{tr}})$ be the section of $\pi_1(X_c^{\mathrm{tr}})$ induced by s. This naturally induces a section $\tilde{s}_c : G_{K_c} \to \pi_1(X_c - \tilde{S})$ of the quotient. The results of §3.4 describe the ramification of $\tilde{s}_c$.

Theorem 4.1.9. *With the notation and assumptions of the above paragraph, fix a universal pro-étale cover $\tilde{X}_{c,\tilde{S}} \to X_c - \tilde{S}$ (see Definition 3.3.2), and denote by $k(\mathcal{X}_c)^{\mathrm{sep}}$ the separable closure of the function field of $\mathcal{X}_c$.*

(i) *Suppose $\tilde{s}_c$ is unramified and induces a section $\bar{s}_c : G_{k(c)} \to G_{\mathcal{X}_c}$. If $\bar{s}_c$ is geometric with $\bar{s}_c(G_{k(c)}) \subset D_{\tilde{x}}$ for some $x \in \mathcal{X}_c(k(c))$ and some extension $\tilde{x}$ of x to $k(\mathcal{X}_c)^{\mathrm{sep}}$, then $\tilde{s}_c(G_{K_c}) \subset D_{\tilde{y}}$ for some $\tilde{y}$ in $\tilde{X}_{c,\tilde{S}}$ above the unique (K_c-rational) point y of $\tilde{S}$ specialising to x.*

(ii) *Assume further that k satisfies condition (ii) of Definition 2.3.1. Suppose $\tilde{s}_c$ is ramified, and denote $\varphi_s := \mathrm{Sp} \circ \tilde{s}_c$. Then $\varphi_s(G_{K_c}) \subset D_{\tilde{x}}$ for a unique valuation on $k(\mathcal{X}_c)^{\mathrm{sep}}$ extending a $k(c)$-rational point x of $\mathcal{X}_c$, and $\tilde{s}_c(G_{K_c}) \subset D_{\tilde{y}}$ for some $\tilde{y}$ in $\tilde{X}_{c,\tilde{S}}$ above the unique (K_c-rational) point y of $\tilde{S}$ specialising to x.*

Proof. Part (i) follows from Lemma 3.4.17, and (ii) follows from Lemma 3.4.16. □

In the following, fix universal pro-étale covers $\tilde{X}_c \to X_c$ and $\tilde{\mathcal{X}}_c \to \mathcal{X}_c$, and denote by $\bar{\eta}_c : \operatorname{Spec} \Omega'_c \to \mathcal{X}_c$ a geometric point with image the generic point of $\mathcal{X}_c$. See Definition 1.4.14 for the definition of the decomposition subgroups.

Corollary 4.1.10. *Assume that k satisfies conditions (ii) and (iii) (a) of Definition 2.3.1. Assume further that the birational section conjecture holds for $\mathcal{X}_c$. Let $s_c^{\text{ét}} : G_{K_c} \to \pi_1(X_c, \bar{\xi}_c)$ denote the (unramified) section of $\pi_1(X_c, \bar{\xi}_c)$ induced by s_c,* *and $\bar{s}_c^{\text{ét}} : G_{k(c)} \to \pi_1(\mathcal{X}_c, \bar{\eta}_c)$ its specialisation. Then $\bar{s}_c^{\text{ét}}(G_{k(c)})$ is contained in a decomposition subgroup $D_{\tilde{x}} \subset \pi_1(\mathcal{X}_c, \bar{\eta}_c)$ for a unique $k(c)$-rational point x of $\mathcal{X}_c$ and some $\tilde{x}$ in $\tilde{\mathcal{X}}_c$ above x, and $s_c^{\text{ét}}(G_{K_c})$ is contained in a decomposition subgroup $D_{\tilde{y}} \subset \pi_1(X_c, \bar{\xi}_c)$ for some $\tilde{y}$ in $\tilde{X}_c$ above the unique (K_c-rational) point y of $\tilde{S}$ specialising to x.*

Proof. Follows from Proposition 3.4.18 (see also its preceding paragraph). □

4.2 Étale abelian sections

We use the notation of §4.1, and hereafter we assume that k has characteristic zero, that X is hyperbolic, and that $X(K) \neq \emptyset$. In this section we explore the étale abelian portions of sections of G_X.

Let $\bar{\xi} : \operatorname{Spec} \Omega \to X$ be a geometric point with image the generic point of X, and similarly let $\bar{\xi}_c' : \operatorname{Spec} \Omega_c' \to \mathcal{X}_c$ be a geometric point of $\mathcal{X}_c$ with image the generic point of $\mathcal{X}_c$. By composition, $\bar{\xi}$ induces a geometric point of $\mathcal{X}$, which we will again denote by $\bar{\xi}$, and a geometric point of C, which we denote by $\bar{\eta}$. It also induces, by the pullback property, a geometric point $\bar{\xi}_c : \operatorname{Spec} \Omega_c \to X_c$ with image the generic point of X_c.

Lemma 4.2.1. *For each $c \in C^{\text{cl}}$, there is a commutative diagram*

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(X_{\overline{K}}, \bar{\xi}) \arrow[r] \arrow[d, equal] & \pi_1(X, \bar{\xi}) \arrow[r] \arrow[d, two heads] & G_K = G_C \arrow[r] \arrow[d, two heads] & 1 \\
1 \arrow[r] & \pi_1(X_{\overline{K}}, \bar{\xi}) \arrow[r] & \pi_1(\mathcal{X}, \bar{\xi}) \arrow[r] & \pi_1(C, \bar{\eta}) \arrow[r] & 1 \\
1 \arrow[r] & \pi_1(\mathcal{X}_{\overline{k(c)}}, \bar{\xi}_c') \arrow[r] \arrow[u, "\wr"'] & \pi_1(\mathcal{X}_c, \bar{\xi}_c') \arrow[r] \arrow[u, hook] & G_{k(c)} \arrow[r] \arrow[u, hook] & 1
\end{tikzcd} \tag{4.3}$$

where the lower vertical homomorphisms are defined up to conjugation.

Proof. This follows from functoriality of the fundamental group and the fundamental exact sequences for X and $\mathcal{X}_c$. Exactness of the middle row follows from [GR71, Exposé XIII, Proposition 4.3], noting that there exists a section of the morphism $\mathcal{X} \to C$. Indeed, since this morphism is proper and C is a smooth curve, the valuative criterion of properness implies that $X(K) = \mathcal{X}(C)$, and $X(K)$ is non-empty by assumption. The lower left vertical map is an isomorphism by Theorem 3.1.4 and Lemma 3.1.2 (ii) (see also the discussion after Corollary 3.1.5). The upper right vertical map is surjective by Lemma 1.4.21, and the lower right vertical map is injective by Proposition 1.4.11. Surjectivity of the upper middle vertical map may be shown via an elementary diagram chase, and similarly for injectivity of the lower middle vertical map. $\square$

A section $s : G_K \to G_X$ induces a section $s^{\text{ét}} : G_K \to \pi_1(X, \bar{\xi})$ of the étale fundamental group of X (see the last two paragraphs of §1.6). By Lemma 4.1.5, the section s also pulls back to a section $s_c : G_{K_c} \to \pi_1(X_c^{\text{tr}})$, and this induces an étale section $s_c^{\text{ét}} : G_{K_c} \to \pi_1(X_c, \bar{\xi}_c)$. By Proposition 3.4.8 (see also Remark 3.4.9), the section $s_c^{\text{ét}}$ specialises to a section $\bar{s}_c^{\text{ét}} : G_{k(c)} \to \pi_1(\mathcal{X}_c, \bar{\xi}_c')$. The following Lemma shows that we may also consider $\bar{s}_c^{\text{ét}}$ the specialisation of $s^{\text{ét}}$.

Lemma 4.2.2. *The étale section $s^{\text{ét}}$ extends to a section $s_C^{\text{ét}} : \pi_1(C, \bar{\eta}) \to \pi_1(\mathcal{X}, \bar{\xi})$ which restricts to the section $s_c^{\text{ét}}$ for each $c \in C^{\text{cl}}$.*

Proof. The étale sections $s^{\text{ét}}$, $s_c^{\text{ét}}$ and $\bar{s}_c^{\text{ét}}$ fit into the following commutative diagram.

$$\begin{tikzcd}
\pi_1(X_c, \bar{\xi}_c) \arrow[rr] \arrow[dr] \arrow[dd, two heads, "\mathrm{Sp}_X"'] & & G_{K_c} \arrow[ll, bend right, "s_c^{\text{ét}}"'] \arrow[dr] \arrow[dd, two heads, "\rho_c" near end] & \\
& \pi_1(X, \bar{\xi}) \arrow[rr] \arrow[dd, two heads] & & G_K = G_C \arrow[ll, bend right, "s^{\text{ét}}"'] \arrow[dd, two heads] \\
\pi_1(\mathcal{X}_c, \bar{\xi}_c') \arrow[rr] \arrow[dr] & & G_{k(c)} \arrow[ll, bend right, "\bar{s}_c^{\text{ét}}"'] \arrow[dr] & \\
& \pi_1(\mathcal{X}, \bar{\xi}) \arrow[rr] & & \pi_1(C, \bar{\eta})
\end{tikzcd}$$

The kernel of the homomorphism $G_C \twoheadrightarrow \pi_1(C, \bar{\eta})$ is the inertia group I_C normally generated by the inertia subgroups associated to the closed points of C, or in other words, by the inertia subgroups $I_{K_c} \subset G_{K_c}$, where each I_{K_c} is the inertia group of the unique valuation of K_c and the kernel of the homomorphism ρ_c.

Since $s_c^{\text{ét}}$ is unramified, the image of I_{K_c} in $\pi_1(\mathcal{X}_c, \bar{\xi}_c')$ is trivial, and since this is true for every $c \in C^{\text{cl}}$ we conclude that the image of I_C in $\pi_1(\mathcal{X}, \bar{\xi})$ is trivial. Thus

$s^{\text{ét}}$ extends to a section $s_C^{\text{ét}} : \pi_1(C, \bar{\eta}) \to \pi_1(\mathcal{X}, \bar{\xi})$, which, by commutativity of the above diagram, restricts to $\bar{s}_c^{\text{ét}} : G_{k(c)} \to \pi_1(\mathcal{X}_c, \bar{\xi}'_c)$. $\square$

Let $\mathcal{J} := \mathrm{Pic}^0_{\mathcal{X}/C} \to C$ denote the relative Jacobian of the relative curve $\mathcal{X}$, and $J := \mathcal{J}_K$ the Jacobian of X (see Remark 1.1.10). For each closed point $c \in C^{\mathrm{cl}}$, let $J_c := J_{K_c}$ denote the Jacobian of X_c and $\mathcal{J}_c := \mathcal{J}_{k(c)}$ that of $\mathcal{X}_c$. An element of $X(K)$ (which is non-empty by assumption) induces a section of the geometrically abelian quotient $\pi_1(X, \bar{\xi})^{(\mathrm{ab})}$ of the étale fundamental group of X, hence, by Theorem 1.4.10 and Theorem 1.5.5, there is a natural identification of the set of conjugacy classes of sections of $\pi_1(X, \bar{\xi})^{(\mathrm{ab})}$ with the cohomology group $H^1(G_K, TJ)$ (see Lemma 1.6.6 and the paragraph after it). Similarly, since $X_c(K_c) \neq \emptyset$ and $\mathcal{X}_c(k(c)) \neq \emptyset$ (indeed, we can take the pullback of an element of $X(K)$ to X_c and its specialisation to $\mathcal{X}_c$), the set of conjugacy classes of sections of $\pi_1(X_c, \bar{\xi}_c)^{(\mathrm{ab})}$, respectively $\pi_1(\mathcal{X}_c, \bar{\xi}'_c)^{(\mathrm{ab})}$, may be identified with $H^1(G_{K_c}, TJ_c)$, resp. $H^1(G_{k(c)}, T\mathcal{J}_c)$.

The above étale sections $s^{\text{ét}}$, $s_c^{\text{ét}}$ and $\bar{s}_c^{\text{ét}}$ induce étale abelian sections s^{ab}, s_c^{ab} and $\bar{s}_c^{\mathrm{ab}}$ respectively, while diagram 4.3 induces a commutative diagram of exact sequences of geometrically abelian fundamental groups

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(X_{\overline{K}}, \bar{\xi})^{\mathrm{ab}} \arrow[r] \arrow[d, equal] & \pi_1(X, \bar{\xi})^{(\mathrm{ab})} \arrow[r] \arrow[d, two heads] & G_K = G_C \arrow[r] \arrow[l, bend right, "s^{\mathrm{ab}}"'] \arrow[d, two heads] & 1 \\
1 \arrow[r] & \pi_1(X_{\overline{K}}, \bar{\xi})^{\mathrm{ab}} \arrow[r] & \pi_1(\mathcal{X}, \bar{\xi})^{(\mathrm{ab})} \arrow[r] & \pi_1(C, \bar{\eta}) \arrow[r] \arrow[l, bend right, "s_C^{\mathrm{ab}}"'] & 1 \\
1 \arrow[r] & \pi_1(\mathcal{X}_{\overline{k(c)}}, \bar{\xi}'_c)^{\mathrm{ab}} \arrow[r] \arrow[u, "\wr"'] & \pi_1(\mathcal{X}_c, \bar{\xi}'_c)^{(\mathrm{ab})} \arrow[r] \arrow[u, hook] & G_{k(c)} \arrow[r] \arrow[l, bend right, "\bar{s}_c^{\mathrm{ab}}"'] \arrow[u, hook] & 1
\end{tikzcd} \tag{4.4}$$

where $\pi_1(\mathcal{X}, \bar{\xi})^{(\mathrm{ab})}$ is defined so that the middle horizontal sequence is exact, and $s_C^{\mathrm{ab}} : \pi_1(C, \bar{\eta}) \to \pi_1(\mathcal{X}, \bar{\xi})^{(\mathrm{ab})}$ is induced by $s_C^{\text{ét}}$. The étale abelian sections s^{ab}, s_c^{ab}, s_C^{ab} and $\bar{s}_c^{\mathrm{ab}}$ are elements of the cohomology groups $H^1(G_K, TJ)$, $H^1(G_{K_c}, TJ_c)$, $H^1(\pi_1(C, \bar{\eta}), TJ)$ and $H^1(G_{k(c)}, T\mathcal{J}_c)$ respectively, which are related by the following restriction and inflation maps:

$$
\begin{tikzcd}
H^1(G_K, TJ) \arrow[r, "\mathrm{res}_c"] & H^1(G_{K_c}, TJ_c) \\
H^1(\pi_1(C,\bar\eta), TJ) \arrow[u, "\mathrm{inf}_C"] \arrow[r, "\mathrm{res}_{C,c}"] & H^1(G_{k(c)}, T\mathcal{J}_c) \arrow[u, "\mathrm{inf}_c"']
\end{tikzcd}
$$

Lemma 4.2.3. *With the above notation, we have the following.*

(i) $\mathrm{res}_c(s^{\mathrm{ab}}) = \mathrm{inf}_c(\bar{s}_c^{\mathrm{ab}}) = s_c^{\mathrm{ab}}$*;*

(ii) $\mathrm{inf}_C(s_C^{\mathrm{ab}}) = s^{\mathrm{ab}}$ *and* $\mathrm{res}_{C,c}(s_C^{\mathrm{ab}}) = \bar{s}_c^{\mathrm{ab}}$.

$$
\begin{tikzcd}
H^1(G_K, TJ) \arrow[rrr, "\mathrm{res}_c"] & & & H^1(G_{K_c}, TJ_c) \\
& s^{\mathrm{ab}} \arrow[r, mapsto] & s_c^{\mathrm{ab}} & \\
& s_C^{\mathrm{ab}} \arrow[u, mapsto] \arrow[r, mapsto] & \bar{s}_c^{\mathrm{ab}} \arrow[u, mapsto] & \\
H^1(\pi_1(C,\bar\eta), TJ) \arrow[uuu, "\mathrm{inf}_C"] \arrow[rrr, "\mathrm{res}_{C,c}"'] & & & H^1(G_{k(c)}, T\mathcal{J}_c) \arrow[uuu, "\mathrm{inf}_c"']
\end{tikzcd}
$$

Proof. Part (i) follows from the fact that $s^{\text{ét}}$ and $\bar{s}_c^{\text{ét}}$ both pull back to $s_c^{\text{ét}}$ - see diagrams 4.1 and 3.11, considering the case when $U = X$ is proper. Part (ii) follows from diagram 4.4. Note that the statement of this Lemma may also be found in [Saï16, Lemma 3.4]. □

The image of s^{ab} under the diagonal map

$$
\prod_{c \in C^{\mathrm{cl}}} \mathrm{res}_c : H^1(G_K, TJ) \longrightarrow \prod_{c \in C^{\mathrm{cl}}} H^1(G_{K_c}, TJ_c)
$$

is therefore the family $(s_c^{\mathrm{ab}})_{c \in C^{\mathrm{cl}}}$. This diagonal map fits into the following commutative diagram of Kummer exact sequences (see Lemma 1.6.7).

$$
\begin{tikzcd}
0 \arrow[r] & \widehat{J(K)} \arrow[r] \arrow[d] & H^1(G_K, TJ) \arrow[r] \arrow[d] & TH^1(G_K, J(\overline{K})) \arrow[r] \arrow[d, "\Phi"] & 0 \\
0 \arrow[r] & \prod_{c \in C^{\mathrm{cl}}} \widehat{J_c(K_c)} \arrow[r] & \prod_{c \in C^{\mathrm{cl}}} H^1(G_{K_c}, TJ_c) \arrow[r] & \prod_{c \in C^{\mathrm{cl}}} TH^1(G_{K_c}, J_c(\overline{K_c})) \arrow[r] & 0
\end{tikzcd}
\tag{4.5}
$$

where $\overline{K}$, respectively $\overline{K_c}$ is the algebraic closure of K in Ω, resp. Ω_c. We denote the right vertical map in this diagram by Φ. See Notation for the meaning of $\widehat{J(K)}$ and $\widehat{J_c(K_c)}$.

Lemma 4.2.4. *The kernel of the map Φ is the Tate module $T\text{Ш}(\mathcal{J})$ of the Shafarevich-Tate group $\text{Ш}(\mathcal{J})$.*

(See Definition 2.3.2 for the definition of the Shafarevich-Tate group.)

Proof. For each $c \in C^{\mathrm{cl}}$ and each positive integer N, the homomorphism $\phi_c : H^1(G_K, J(\overline{K})) \to H^1(G_{K_c}, J_c(\overline{K_c}))$ induces a homomorphism of N-torsion groups $\phi_{c,N} : H^1(G_K, J(\overline{K}))[N] \to H^1(G_{K_c}, J_c(\overline{K_c}))[N]$ (see Notation). Thus the diagonal map $\prod_{c\in C^{\mathrm{cl}}} \phi_c$ induces a map

$$\Phi_N := \prod_{c\in C^{\mathrm{cl}}} \phi_{c,N} : H^1(G_K, J(\overline{K}))[N] \to \prod_{c\in C^{\mathrm{cl}}} H^1(G_{K_c}, J_c(\overline{K_c}))[N]$$

The kernel of this map is $\text{Ш}(\mathcal{J})[N]$. Taking the projective limit over N, we find that the kernel of the map

$$\Phi = \varprojlim_N \Phi_N : TH^1(G_K, J(\overline{K})) \to \prod_{c\in C^{\mathrm{cl}}} TH^1(G_{K_c}, J_c(\overline{K_c}))$$

is $\varprojlim_N \text{Ш}(\mathcal{J})[N] = T\text{Ш}(\mathcal{J})$, since the projective limit $\varprojlim_N$ is left exact. $\square$

Proposition 4.2.5. *Suppose that, for every $c \in C^{\mathrm{cl}}$, the section $s_c^{\mathrm{ab}} \in H^1(G_{K_c}, TJ_c)$ is contained in $\widehat{J_c(K_c)}$. Then if $T\text{Ш}(\mathcal{J}) = 0$, the section $s^{\mathrm{ab}} \in H^1(G_K, TJ)$ is contained in $\widehat{J(K)}$.*

Proof. By commutativity of diagram 4.5 and the assumption that $s_c^{\mathrm{ab}} \in \widehat{J_c(K_c)}$ for every $c \in C^{\mathrm{cl}}$, the image of s^{ab} in $TH^1(G_K, J(\overline{K}))$ must be contained in $\ker\Phi = T\text{Ш}(\mathcal{J})$. Since this is trivial, s^{ab} must map trivially to $TH^1(G_K, J(\overline{K}))$, hence $s^{\mathrm{ab}} \in \widehat{J(K)}$. $\square$

Corollary 4.2.6. *Assume that $T\text{Ш}(\mathcal{J}) = 0$, and that k strongly satisfies conditions (i) and (ii) of Definition 2.3.1. Then s^{ab} is contained in $\widehat{J(K)}$.*

Proof. Under these conditions on k, Theorem 4.1.9 implies that, for every $c \in C^{\mathrm{cl}}$, the section $\tilde{s}_c : G_{K_c} \to \pi_1(X_c - \tilde{S})$ satisfies $\tilde{s}_c(G_{K_c}) \subset D_{\tilde{x}_c}$ for some K_c-rational point x_c of X_c and some $\tilde{x}_c$ above x_c in a universal pro-étale cover $\tilde{X}_{c,\tilde{S}} \to X_c -$

$\tilde{S}$. Thus $s_c^{\text{ét}}(G_{K_c}) = D_{\tilde{x}'_c} \subset \pi_1(X_c, \bar{\xi}_c)$, for some $\tilde{x}'_c$ above x_c in a universal pro-étale cover $\tilde{X}_c \to X_c$, which implies that s_c^{ab} is the image of x_c under the map $X_c(K_c) \to H^1(G_{K_c}, TJ_c)$ (see the paragraph before Lemma 1.6.7). Since this map factors through $\widehat{J_c(K_c)}$ (see sequence 1.7 in §1.6 and the discussion before and after Lemma 1.6.7), this implies that s_c^{ab} is the image of x_c in the composite map

$$X_c(K_c) \hookrightarrow J_c(K_c) \to \widehat{J_c(K_c)} \hookrightarrow H^1(G_{K_c}, TJ_c)$$

In particular, s_c^{ab} is contained in the image of $\widehat{J_c(K_c)}$, and since this is true for every $c \in C^{\text{cl}}$, Proposition 4.2.5 implies that $s^{\text{ab}} \in \widehat{J(K)}$. $\square$

Lemma 4.2.7. *Assume that $k(c)$ satisfies conditions (ii) and (iii) (a) of Definition 2.3.1, and assume the birational section conjecture holds for $\mathcal{X}_c$. Then $\bar{s}_c^{\text{ab}}$ is in the image of the map $\mathcal{X}_c(k(c)) \to H^1(G_{k(c)}, T\mathcal{J}_c)$, and s_c^{ab} is in the image of $X_c(K_c) \to H^1(G_{K_c}, TJ_c)$.*

Proof. This follows from Corollary 4.1.10, since $\bar{s}_c^{\text{ét}}(G_{k(c)}) \subset D_{\tilde{x}}$ implies that $\bar{s}_c^{\text{ét}}$ arises from $x \in \mathcal{X}_c(k(c))$ by functoriality, and likewise $s_c^{\text{ét}}(G_{K_c}) \subset D_{\tilde{y}}$ implies $s_c^{\text{ét}}$ arises from $y \in X_c(K_c)$ (see also the paragraph before Lemma 1.6.7). $\square$

Chapter 5

Proof of the Main Theorems

In this chapter we prove Theorems A and B.

5.1 Proof of Theorem A

We use the notation from Theorem A. Let k be a field of characteristic zero that strongly satisfies the conditions of Definition 2.3.1. Let C be a smooth, separated, connected curve over k with function field K. Let $\mathcal{X} \to C$ be a smooth relative curve whose generic fibre $X := \mathcal{X} \times_C \operatorname{Spec} K$ is geometrically connected and hyperbolic, with $X(K) \neq \emptyset$. Note that since the morphism $\mathcal{X} \to C$ is flat, hyperbolicity of X implies that, for every $c \in C^{\mathrm{cl}}$, the closed fibre $\mathcal{X}_c := \mathcal{X} \times_C \operatorname{Spec} k(c)$ is hyperbolic. Let $\mathcal{J} := \operatorname{Pic}^0_{\mathcal{X}/C}$ denote the relative Jacobian of $\mathcal{X}$, and $J := \mathcal{J}_K$ the Jacobian of X. For a closed point $c \in C^{\mathrm{cl}}$, denote by $\mathcal{J}_c := \mathcal{J}_{k(c)}$ the Jacobian of $\mathcal{X}_c$. Assume that $T\text{Ш}(\mathcal{J}) = 0$.

We will show that the birational section conjecture holds for X, which is to say that every section of the projection $G_X \twoheadrightarrow G_K$ arises from a unique K-rational point of X (see Definition 2.2.2). It suffices to show the existence of such a point, since its uniqueness follows from [NSW08, Corollary 12.1.3] (see Remark 2.2.3). The proofs in this section are inspired by the proofs in [Saï16, §4.1] - the proofs of Lemmas 5.1.1 to 5.1.3 in particular are directly comparable to those of Lemmas 4.1.1 to 4.1.3 in [Saï16].

Let $s : G_K \to G_X$ be any section. We must show that $s(G_K)$ is contained in a decomposition subgroup $D_{\tilde{x}} \subset G_X$ for some K-rational point x of X and some extension $\tilde{x}$ of x to $k(X)^{\mathrm{sep}}$. Under our assumptions, Corollary 4.2.6 implies that the abelian portion s^{ab} of s is contained in $\widehat{J(K)}$.

Lemma 5.1.1. *The homomorphism $J(K) \to \widehat{J(K)}$ is injective and s^{ab} is contained in $J(K)$.*

Proof. There exist $c_1, c_2 \in C^{\mathrm{cl}}$ such that the natural specialisation homomorphism $J(K) \to \mathcal{J}_{c_1}(k(c_1)) \times \mathcal{J}_{c_2}(k(c_2))$ is injective [PV10, Proposition 2.4]. Let l be a finite extension of k that contains $k(c_1)$ and $k(c_2)$. Then there is an injective homomorphism $\mathcal{J}_{c_i}(k(c_i)) \hookrightarrow \mathcal{J}_{c_i}(l)$ for each $i = 1, 2$, and these induce an injective homomorphism $\mathcal{J}_{c_1}(k(c_1)) \times \mathcal{J}_{c_2}(k(c_2)) \hookrightarrow \mathcal{J}_{c_1}(l) \times \mathcal{J}_{c_2}(l) \simeq (\mathcal{J}_{c_1} \times \mathcal{J}_{c_2})(l)$. For ease of notation, let us write $\mathcal{J}_{1,2}(l) := (\mathcal{J}_{c_1} \times \mathcal{J}_{c_2})(l)$. Denoting $H := \mathcal{J}_{1,2}(l)/J(K)$, we thus have an exact sequence:

$$0 \longrightarrow J(K) \longrightarrow \mathcal{J}_{1,2}(l) \longrightarrow H \longrightarrow 0$$

The multiplication by N map on these abelian groups induces an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & J(K)[N] \arrow[r] & \mathcal{J}_{1,2}(l)[N] \arrow[r] & H[N] \arrow[dll, "\delta_N"] & \\
& \dfrac{J(K)}{NJ(K)} \arrow[r, "\alpha_N"] & \dfrac{\mathcal{J}_{1,2}(l)}{N\mathcal{J}_{1,2}(l)} \arrow[r, "\beta_N"] & \dfrac{H}{NH} \arrow[r] & 0
\end{tikzcd} \tag{5.1}$$

where the map δ_N comes from the Snake Lemma. By condition (iii)(b) of Definition 2.3.1 we have $\varprojlim_N H[N] = TH = 0$, thus, taking the inverse limit of sequence 5.1 as $N \to \infty$, we see that the homomorphism

$$\varprojlim_N \alpha_N : \widehat{J(K)} \to \widehat{\mathcal{J}_{1,2}(l)}$$

is injective. Moreover, sequence 5.1 induces a short exact sequence

$$0 \longrightarrow \frac{J(K)/NJ(K)}{\delta_N(H[N])} \xrightarrow{\alpha_N} \frac{\mathcal{J}_{1,2}(l)}{N\mathcal{J}_{1,2}(l)} \xrightarrow{\beta_N} \frac{H}{NH} \longrightarrow 0 \tag{5.2}$$

Since the maps $J(K)/MJ(K) \to J(K)/NJ(K)$ is surjective for any $N|M$, the maps

$$\frac{J(K)/MJ(K)}{\delta_M(H[M])} \longrightarrow \frac{J(K)/NJ(K)}{\delta_N(H[N])} \qquad \text{for } N|M$$

are surjective, hence the inverse system they define satisfies the Mittag-Leffler condition. Therefore, taking the projective limit of sequence 5.2 as $N \to \infty$, we find that the homomorphism

$$\varprojlim_N \beta_N : \widehat{\mathcal{J}_{1,2}(l)} \longrightarrow \widehat{H}$$

is surjective. Thus we have a commutative diagram of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & J(K) \arrow[r] \arrow[d] & \mathcal{J}_{1,2}(l) \arrow[r] \arrow[d, "\phi"'] & H \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \widehat{J(K)} \arrow[r, "\psi"] & \widehat{\mathcal{J}_{1,2}(l)} \arrow[r] & \widehat{H} \arrow[r] & 0
\end{tikzcd}$$

The middle and right vertical maps are injective by condition (iii)(a) of Definition 2.3.1, and this implies that the left vertical map is injective. Therefore the equality $J(K) = \phi(\mathcal{J}_{1,2}(l)) \cap \psi(\widehat{J(K)})$ holds inside $\widehat{\mathcal{J}_{1,2}(l)}$.

For each c_i, $i = 1, 2$, the section s induces an element $\bar{s}_{c_i}^{\mathrm{ab}} \in H^1(G_{k(c_i)}, T\mathcal{J}_{c_i})$ (see the discussion after Lemma 4.2.1), which is contained in the image of the map $\mathcal{X}_{c_i}(k(c_i)) \to H^1(G_{k(c_i)}, T\mathcal{J}_{c_i})$ by Lemma 4.2.7. This map is injective by condition (iii)(a) of Definition 2.3.1, so we may consider $\bar{s}_{c_i}^{\mathrm{ab}}$ to be contained in $\mathcal{X}_{c_i}(k(c_i))$. Then $\bar{s}_{c_i}^{\mathrm{ab}}$ is contained in $\mathcal{J}_{c_i}(l)$ for each $i = 1, 2$, due to injectivity of the maps $\mathcal{X}_{c_i}(k(c_i)) \hookrightarrow \mathcal{J}_{c_i}(k(c_i)) \hookrightarrow \mathcal{J}_{c_i}(l)$. Thus $(\bar{s}_{c_1}^{\mathrm{ab}}, \bar{s}_{c_2}^{\mathrm{ab}})$ is contained in $\mathcal{J}_{c_1}(l) \times \mathcal{J}_{c_2}(l)$, hence in $\phi(\mathcal{J}_{1,2}(l))$. By Lemma 4.2.3, the image of $s^{\mathrm{ab}} \in \widehat{J(K)}$ in $\widehat{\mathcal{J}_{1,2}(l)}$ under ψ is the element $(\bar{s}_{c_1}^{\mathrm{ab}}, \bar{s}_{c_2}^{\mathrm{ab}})$, and since this lies in $\phi(\mathcal{J}_{1,2}(l))$ we have $s^{\mathrm{ab}} \in \phi(\mathcal{J}_{1,2}(l)) \cap \psi(\widehat{J(K)}) = J(K)$. □

Since the morphism $\mathcal{X} \to C$ is proper and C is a smooth curve, the valuative criterion of properness implies that $X(K) = \mathcal{X}(C)$. Fix a K-rational point $x_0 \in X(K) = \mathcal{X}(C)$ (non-empty by assumption), and let $\iota_K : X \to J$ denote the unique closed immersion taking x_0 to 0. This extends uniquely to a closed immersion $\iota : \mathcal{X} \to \mathcal{J}$ by the Néron mapping property (see Remark 1.1.10).

Lemma 5.1.2. *s^{ab} is contained in $X(K)$.*

Proof. Since $\mathcal{J} \to C$ is proper and C is smooth, we have $J(K) = \mathcal{J}(C)$. Since s^{ab} is contained in $J(K)$, we may consider it a morphism $s^{\mathrm{ab}} : C \to \mathcal{J}$. By Lemma 4.2.3, the pullback of this morphism to $\operatorname{Spec} k(c)$ is precisely $\bar{s}_c^{\mathrm{ab}}$, considered an element of $\mathcal{J}_c(k(c)) \subset H^1(G_{k(c)}, T\mathcal{J}_c)$. But $\bar{s}_c^{\mathrm{ab}}$ is contained in $\mathcal{X}_c(k(c))$ by Lemma 4.2.7, hence the morphism $\bar{s}_c^{\mathrm{ab}} : \operatorname{Spec} k(c) \to \mathcal{J}_c$ must factor through $\mathcal{X}_c$, where $\mathcal{X}_c$ is considered a closed subscheme of $\mathcal{J}_c$ via ι. Thus, for each $c \in C^{\mathrm{cl}}$, the image of c under the morphism $s^{\mathrm{ab}} : C \to \mathcal{J}$ is a closed point of $\mathcal{X}_c$. This implies that $s^{\mathrm{ab}} : C \to \mathcal{J}$ factors through $\iota : \mathcal{X} \to \mathcal{J}$, thus s^{ab} is contained in the subset $\iota(X(K)) \subseteq J(K)$. $\square$

Let z denote the point in $X(K)$ such that $\iota(z) = s^{\mathrm{ab}}$, and for each $c \in C^{\mathrm{cl}}$ let $\bar{z}_c$ denote its specialisation to $\mathcal{X}_c$. Let $\bar{x}_c \in \mathcal{X}_c(k(c))$ be the point associated to $\bar{s}_c^{\mathrm{ab}}$ by the assumption that the birational section conjecture holds over $k(c)$ (see Lemma 4.2.7).

Lemma 5.1.3. *$\bar{z}_c = \bar{x}_c$ in $\mathcal{X}_c(k(c))$ for all $c \in C^{\mathrm{cl}}$.*

Proof. Lemma 4.2.3 implies that, for each $c \in C^{\mathrm{cl}}$, $\bar{s}_c^{\mathrm{ab}}$ is the image of both $\bar{x}_c$ and $\bar{z}_c$ under the composite map $\mathcal{X}_c(k(c)) \hookrightarrow \mathcal{J}_c(k(c)) \to \widehat{\mathcal{J}_c(k(c))} \hookrightarrow H^1(G_{k(c)}, T\mathcal{J}_c)$. Condition (iii)(a) of Definition 2.3.1 implies this composite map is injective for each $c \in C^{\mathrm{cl}}$, hence $\bar{z}_c = \bar{x}_c$. $\square$

Proposition 5.1.4. *s is geometric.*

Proof. Recall the definition of *neighbourhood* from Definition 2.2.4: a neighbourhood of the section $s : G_K \to G_X$ is an open subgroup $H \subset G_X$ which contains $s(G_K)$. For such a neighbourhood H, if $X_H \to X$ denotes the finite morphism corresponding to H, we have $G_{X_H} = H$. By the "limit argument" of Lemma 2.2.5, to prove the Proposition it suffices to show that every neighbourhood H of s satisfies $X_H(K) \neq \emptyset$.

So let H be a neighbourhood of s, and let $f : Y \to X$ denote the corresponding finite morphism, so that $G_Y = H$. We will show that $Y(K) \neq \emptyset$. By definition, s defines a section of the subgroup $G_Y \subset G_X$, which we shall call s_Y.

$$\begin{tikzcd} G_Y \arrow[r, hook] & G_X \arrow[r] & G_K \arrow[l, bend right, "s"'] \arrow[ll, bend right, "s_Y"'] \end{tikzcd}$$

For $c \in C^{\mathrm{cl}}$, let K_c denote the completion of K at c, and denote $X_c := X \times_{\operatorname{Spec} K} \operatorname{Spec} K_c$ and $Y_c := Y \times_{\operatorname{Spec} K} \operatorname{Spec} K_c$. Let f_c denote the morphism $f_c : Y_c \to X_c$. By Lemma 4.1.5, the section $s : G_K \to G_X$ pulls back to a section $s_c : G_{K_c} \to \pi_1(X_c^{\mathrm{tr}})$ (see Definition 4.1.2 for the definition of $\pi_1(X_c^{\mathrm{tr}})$). Likewise s_Y pulls back to a section

$s_{Y_c} : G_{K_c} \to \pi_1(Y_c^{\mathrm{tr}})$.

$$\begin{tikzcd}
\pi_1(Y_c^{\mathrm{tr}}) \arrow[r, "g"] \arrow[d] & \pi_1(X_c^{\mathrm{tr}}) \arrow[r, "r"'] \arrow[d, "p"] & G_{K_c} \arrow[d] \arrow[l, bend right, "s_c"'] \arrow[ll, bend right, "s_{Y_c}"'] \\
G_Y \arrow[r] & G_X \arrow[r] & G_K \arrow[l, bend right, "s"'] \arrow[ll, bend right, "s_Y"']
\end{tikzcd}$$

Note the existence of g and the fact that the projection $\pi_1(Y_c^{\mathrm{tr}}) \twoheadrightarrow G_{K_c}$ coincides with $r \circ g$ follow from the universal property of pullbacks.

Lemma 5.1.5. *With notation as in the above diagram, $g(s_{Y_c}(G_{K_c})) = s_c(G_{K_c})$.*

Proof. Take $\sigma \in G_{K_c}$, and consider $s_c(\sigma)$ and $g(s_{Y_c}(\sigma))$. Since s_{Y_c} is the pullback of s_Y and s_c is the pullback of s, we must have $p(s_c(\sigma)) = p(g(s_{Y_c}(\sigma)))$ in G_X. If $s_c(\sigma)$ and $g(s_{Y_c}(\sigma))$ are not equal then $g(s_{Y_c}(\sigma)) = \tau \cdot s_c(\sigma)$ for some $\tau \in \ker(p)$. Then:

$$\begin{aligned}
r(g(s_{Y_c}(\sigma))) &= r(\tau \cdot s_c(\sigma)) \\
&= r(\tau) \cdot r(s_c(\sigma)) \\
&= r(\tau) \cdot \sigma
\end{aligned}$$

But since the projection $\pi_1(Y_c^{\mathrm{tr}}) \twoheadrightarrow G_{K_c}$ coincides with $r \circ g$, we have $r(g(s_{Y_c}(\sigma))) = r(s_c(\sigma)) = \sigma$, and therefore $r(\tau) = 1$ and $\tau \in \ker(r)$. Thus $\tau = 1$, since existence of a non-trivial element of $\ker(p) \cap \ker(r)$ violates the universal property of pullbacks. □

Remark 5.1.6. One can see from the above diagram that the preimage in Y_c of the set of algebraic points of X_c is exactly the set of algebraic points of Y_c. This is because the finite morphism $Y_c^{\mathrm{tr}} \to X_c^{\mathrm{tr}}$ is étale by construction, hence any finite morphism $Z \to Y_c$ which is étale over Y_c^{tr} comes from a finite morphism $Z' \to X_c$ which is étale over X_c^{tr} (one can take $Z' \to X_c$ to be the composite $Z \to Y_c \to X_c$). This implies that any such $Z \to Y_c$ must be étale over the complement of the preimage in Y_c of the set of algebraic points of X_c, since this is true of the pullback $Z' \times_{X_c} Y_c \to Y_c$. We therefore have $\pi_1(Y_c^{\mathrm{tr}}) = \pi_1(Y_c - f_c^{-1}(\text{alg. pts}))$, which proves the assertion.

Let $K(Y)$ denote the function field of Y, and let $\mathcal{Y}$ denote the normalisation of $\mathcal{X}$ in $K(Y)$. Thus $\mathcal{Y}$ is a normal relative curve over C with generic fibre Y, and the normalisation morphism $f : \mathcal{Y} \to \mathcal{X}$ extends the map $Y \to X$. Let f_K denote the finite morphism $f_K : Y \to X$, and $\bar{f}_c$ the finite morphism $\mathcal{Y}_c \to \mathcal{X}_c$, where $\mathcal{Y}_c$ is the closed fibre of $\mathcal{Y}$ at $c \in C^{\mathrm{cl}}$.

After possibly removing finitely many points from C, we may assume that $\mathcal{Y}$ is smooth over C. Indeed, by [Liu02, Proposition 10.1.21] the closed fibres $\mathcal{Y}_c$ of $\mathcal{Y}$ are smooth except possibly for finitely many closed points $c \in C^{\mathrm{cl}}$. So, if necessary, we may replace C by the largest open subset $C' \subset C$ such that $\mathcal{Y}_c$ is smooth for every $c \in (C')^{\mathrm{cl}}$, and f by the induced map of fibre products $f \times_C C' : \mathcal{Y} \times_C C' \to \mathcal{X} \times_C C'$.

So we assume that the fibres $\mathcal{Y}_c$ are smooth for all $c \in C^{\mathrm{cl}}$. For each such c we have the following commutative diagram.

$$\begin{tikzcd}
Y \arrow[r, "f_K"] \arrow[d] & X \arrow[r] \arrow[d] & \operatorname{Spec} K \arrow[d] \\
\mathcal{Y} \arrow[r, "f"] & \mathcal{X} \arrow[r] & C \\
\mathcal{Y}_c \arrow[r, "\bar{f}_c"] \arrow[u] & \mathcal{X}_c \arrow[r] \arrow[u] & \operatorname{Spec} k(c) \arrow[u]
\end{tikzcd}$$

For $c \in C^{\mathrm{cl}}$, let $\hat{\mathcal{O}}_{C,c}$ denote the valuation ring of K_c. Denote by $\hat{\mathcal{X}}_c := \mathcal{X} \times_C \operatorname{Spec} \hat{\mathcal{O}}_{C,c}$ the base change of $\mathcal{X}$ to $\hat{\mathcal{O}}_{C,c}$, and similarly write $\hat{\mathcal{Y}}_c := \mathcal{Y} \times_C \operatorname{Spec} \hat{\mathcal{O}}_{C,c}$. Let $\hat{f}_c$ denote the morphism $\hat{f}_c : \hat{\mathcal{Y}}_c \to \hat{\mathcal{X}}_c$. We have the following commutative diagram.

$$\begin{tikzcd}
Y_c \arrow[r, "f_c"] \arrow[d] & X_c \arrow[r] \arrow[d] & \operatorname{Spec} K_c \arrow[d] \\
\hat{\mathcal{Y}}_c \arrow[r, "\hat{f}_c"] & \hat{\mathcal{X}}_c \arrow[r] & \operatorname{Spec} \hat{\mathcal{O}}_{C,c} \\
\mathcal{Y}_c \arrow[r, "\bar{f}_c"] \arrow[u] & \mathcal{X}_c \arrow[r] \arrow[u] & \operatorname{Spec} k(c) \arrow[u]
\end{tikzcd}$$

Let $S_c \subset X_c$ denote the finite set of closed points over which f_c is ramified. For each closed point $\bar{x}_c$ of $\mathcal{X}_c$, choose, as in Definition 4.1.7, an algebraic point $x_c \in X_c - S_c$ specialising to $\bar{x}_c$ whose residue field is the unique unramified extension L_c of K_c

whose valuation ring has residue field $k(\bar{x}_c)$. Note it is possible to choose x_c such that $x_c \notin S_c$ since, as shown in the proof of Lemma 4.1.6, choosing an L_c-rational algebraic point of X_c specialising to $\bar{x}_c$ amounts to choosing an element of $\mathfrak{m}_{L_c} \cap L$ for some finite extension L of K which is unramified with respect to (the valuation corresponding to) c. The set $\mathfrak{m}_{L_c} \cap L$ is infinite, hence, since S_c is finite, there are infinitely many possible choices for x_c with $x_c \notin S_c$.

Let $\tilde{S}$ denote this set of chosen points of X_c. Thus, $\tilde{S}$ is a set of algebraic points of X_c in bijection with the closed points of $\mathcal{X}_c$. By Lemma 3.2.1, for any finite subset $D \subset \tilde{S}$ the closure of D in $\hat{\mathcal{X}}_c$ is a divisor on $\hat{\mathcal{X}}_c$ which is finite étale over $\operatorname{Spec} \hat{\mathcal{O}}_{C,c}$. Let $\tilde{T} := f_c^{-1}(\tilde{S})$ denote the preimage of $\tilde{S}$ in Y_c, and similarly $E := f_c^{-1}(D) \subset \tilde{T}$ the preimage of D in Y_c. For each such D there is a morphism of K_c-schemes $Y_c - E \to X_c - D$, which induces a homomorphism $\pi_1(Y_c - E, \bar{\xi}_{Y_c}) \to \pi_1(X_c - D, \bar{\xi}_{X_c})$ for geometric points $\bar{\xi}_{Y_c}$, $\bar{\xi}_{X_c}$ with images the generic points of Y_c, X_c. In the limit as D ranges over the finite subsets of $\tilde{S}$, we obtain a homomorphism

$$\pi_1(Y_c - \tilde{T}) \longrightarrow \pi_1(X_c - \tilde{S})$$

Since $\tilde{S} \subset X_c - S_c$ and f_c is étale over $X_c - S_c$, for each finite subset $D \subset \tilde{S}$ the closure in $\hat{\mathcal{Y}}_c$ of the preimage $E := f_c^{-1}(D) \subset \tilde{T}$ is a divisor on $\hat{\mathcal{Y}}_c$ which is finite étale over $\operatorname{Spec} \hat{\mathcal{O}}_{C,c}$. Moreover, E consists of algebraic points of Y_c (see Remark 5.1.6).

Thus $\tilde{T}$ is a set of algebraic points of Y_c in bijection with the set of closed points of $\mathcal{Y}_c$, and for every finite subset $E \subset \tilde{T}$ the closure of E in $\hat{\mathcal{Y}}_c$ is a divisor on $\hat{\mathcal{Y}}_c$ which is étale over $\operatorname{Spec} \hat{\mathcal{O}}_{C,c}$. Therefore, as in Theorem 4.1.8, there exist specialisation homomorphisms Sp_X, Sp_Y making the following diagram commute.

$$\begin{tikzcd}
\pi_1(Y_c - \tilde{T}) \arrow[r] \arrow[d, two heads, "\mathrm{Sp}_Y"'] & \pi_1(X_c - \tilde{S}) \arrow[r] \arrow[d, two heads, "\mathrm{Sp}_X"'] & G_{K_c} \arrow[d, two heads, "\rho"'] \\
G_{\mathcal{Y}_c} \arrow[r] & G_{\mathcal{X}_c} \arrow[r] & G_{k(c)}
\end{tikzcd}$$

Since $\tilde{T}$ and $\tilde{S}$ consist of algebraic points, the groups $\pi_1(Y_c - \tilde{T})$ and $\pi_1(X_c - \tilde{S})$ are naturally quotients of $\pi_1(Y_c^{\mathrm{tr}})$ and $\pi_1(X_c^{\mathrm{tr}})$ respectively. Thus the section s_c naturally induces a section $\tilde{s}_c : G_{K_c} \to \pi_1(X_c - \tilde{S})$, and likewise s_{Y_c} induces a section $\tilde{s}_{Y_c} : G_{K_c} \to \pi_1(Y_c - \tilde{T})$. We denote $\varphi_X := \mathrm{Sp}_X \circ \tilde{s}_c$ and $\varphi_Y := \mathrm{Sp}_Y \circ \tilde{s}_{Y_c}$.

$$\begin{tikzcd}
\pi_1(Y_c^{\mathrm{tr}}) \arrow[r, "g"] \arrow[d, two heads] & \pi_1(X_c^{\mathrm{tr}}) \arrow[r] \arrow[d, two heads] & G_{K_c} \arrow[l, bend right, "s_c"'] \arrow[ll, bend right=40, "s_{Y_c}"'] \arrow[d, equal] \\
\pi_1(Y_c - \tilde{T}) \arrow[r, "\tilde{g}"] \arrow[d, two heads, "\mathrm{Sp}_Y"'] & \pi_1(X_c - \tilde{S}) \arrow[r] \arrow[d, two heads, "\mathrm{Sp}_X"'] & G_{K_c} \arrow[l, bend right, "\tilde{s}_c"'] \arrow[ll, bend right=40, "\tilde{s}_{Y_c}"'] \arrow[d, two heads, "\rho"] \arrow[dll, two heads, bend left, "\varphi_Y"] \arrow[dl, two heads, bend left, "\varphi_X"] \\
G_{\mathcal{Y}_c} \arrow[r, hook, "\bar{g}"] & G_{\mathcal{X}_c} \arrow[r] & G_{k(c)}
\end{tikzcd}$$

Note that the map $G_{\mathcal{Y}_c} \to G_{\mathcal{X}_c}$ in the above diagram is injective, since $\mathcal{Y}_c \to \mathcal{X}_c$ is a finite morphism.

Lemma 5.1.7. *With notation as in the above diagram, $\tilde{g}(\tilde{s}_{Y_c}(G_{K_c})) = \tilde{s}_c(G_{K_c})$ and $\bar{g}(\varphi_Y(G_{K_c})) = \varphi_X(G_{K_c})$.*

Proof. Follows from Lemma 5.1.5 and commutativity of the above diagram, together with the fact that, as in Lemma 3.4.2, $\ker \mathrm{Sp}_Y$ and $\ker \mathrm{Sp}_X$ both map isomorphically to $\ker \rho \subset G_{K_c}$, and therefore $\tilde{g}(\ker \mathrm{Sp}_Y) = \ker \mathrm{Sp}_X$. ☐

Corollary 5.1.8. *The section $\tilde{s}_c$ is unramified if and only if $\tilde{s}_{Y_c}$ is unramified.*

Proof. By Lemma 5.1.7, $\bar{g}(\varphi_Y(I_{K_c})) = \varphi_X(I_{K_c})$. Since $\bar{g}$ is injective, $\varphi_Y(I_{K_c})$ is trivial if and only if $\varphi_X(I_{K_c})$ is. ☐

Now we consider two cases - the case when $\tilde{s}_c$ and $\tilde{s}_{Y_c}$ are unramified, and the case when they are ramified.

Case 1. Suppose $\tilde{s}_c$ and $\tilde{s}_{Y_c}$ are unramified. Then they induce sections $\bar{s}_c : G_{k(c)} \to G_{\mathcal{X}_c}$ and $\bar{s}_{\mathcal{Y}_c} : G_{k(c)} \to G_{\mathcal{Y}_c}$.

$$
\begin{tikzcd}
\pi_1(Y_c - \tilde{T}) \arrow[r] \arrow[d, two heads, "\mathrm{Sp}_Y"'] & \pi_1(X_c - \tilde{S}) \arrow[r] \arrow[d, two heads, "\mathrm{Sp}_X"'] & G_{K_c} \arrow[d, two heads, "\rho"] \arrow[ll, bend right, "\tilde{s}_{Y_c}"'] \arrow[l, bend right, "\tilde{s}_c"'] \arrow[dll, bend right, two heads, "\varphi_Y"] \arrow[dl, bend right, two heads, "\varphi_X"'] \\
G_{\mathcal{Y}_c} \arrow[r, hook, "\bar{g}"'] & G_{\mathcal{X}_c} \arrow[r] & G_{k(c)} \arrow[ll, bend right, "\bar{s}_{\mathcal{Y}_c}"'] \arrow[l, bend right, "\bar{s}_c"']
\end{tikzcd}
$$

By the assumption of the birational section conjecture over $k(c)$, the sections $\bar{s}_c$ and $\bar{s}_{\mathcal{Y}_c}$ are geometric. Let $\bar{y}_c$ be the $k(c)$-rational point and $\tilde{y}_c$ the extension of $\bar{y}_c$ to $k(\mathcal{X}_c)^{\mathrm{sep}}$ such that $\bar{s}_{\mathcal{Y}_c}(G_{k(c)}) \subset D_{\tilde{y}_c} \subset G_{\mathcal{Y}_c}$. By Lemma 5.1.7 and injectivity of $\tilde{g}$, this implies that $\bar{s}_c(G_{k(c)})$ is contained in the decomposition subgroup $D_{\tilde{y}_c} \subset G_{\mathcal{X}_c}$ of the same valuation $\tilde{y}_c$ on $k(\mathcal{X}_c)^{\mathrm{sep}}$, and the restriction of of $\tilde{y}_c$ to $k(\mathcal{X}_c)$ corresponds to the image of $\bar{y}_c$ in $\mathcal{X}_c$, which we will denote $\bar{x}'_c$.

Case 2. Suppose $\bar{s}_c$ and $\bar{s}_{\mathcal{Y}_c}$ are ramified. Then, by Proposition 3.4.8, Lemma 3.4.14 and Proposition 3.4.15, $\varphi_Y(I_{K_c})$ is contained in the inertia subgroup $I_{\tilde{y}_c} \subset G_{\mathcal{Y}_c}$ of a unique valuation $\tilde{y}_c$ on $k(\mathcal{X}_c)^{\mathrm{sep}}$ extending a $k(c)$-rational point $\bar{y}_c \in \mathcal{Y}_c(k(c))$. By Lemma 5.1.7 and injectivity of $\tilde{g}$, $\varphi_X(I_{K_c})$ is contained in the inertia subgroup $I_{\tilde{y}_c} \subset G_{\mathcal{X}_c}$ of the same valuation $\tilde{y}_c$ on $k(\mathcal{X}_c)^{\mathrm{sep}}$, whose restriction to $k(\mathcal{X}_c)$ corresponds to the image $\bar{x}'_c$ of $\bar{y}_c$ in $\mathcal{X}_c$.

Thus we have found, for every $c \in C^{\mathrm{cl}}$, unique $k(c)$-rational points $\bar{y}_c \in \mathcal{Y}_c(k(c))$ and $\bar{x}'_c \in \mathcal{X}_c(k(c))$ such that $\bar{y}_c$ maps to $\bar{x}'_c$ via $\mathcal{Y}_c \to \mathcal{X}_c$. Moreover, this $\bar{x}'_c$ must be the same as the point $\bar{x}_c$ associated to $\bar{s}_c^{\mathrm{ab}}$ (see Lemma 5.1.3 and the paragraph before it).

Recall the section s^{ab} is associated to a K-rational point z (see Lemma 5.1.2 and the paragraph after it). View $z \in X(K) = \mathcal{X}(C)$ as a section $z : C \to \mathcal{X}$, and denote by $\mathcal{Y}_z$ the pullback of the image $z(C)$ via the map $\mathcal{Y} \to \mathcal{X}$. Then $\mathcal{Y}_z \to z(C)$ is a finite morphism, and since z specialises to $\bar{x}_c \in \mathcal{X}_c(k(c))$ (Lemma 5.1.3), $\bar{x}_c \in z(C)$ and therefore $\bar{y}_c \in \mathcal{Y}_z(k(c))$ for every $c \in C^{\mathrm{cl}}$. Then condition (v) of Definition 2.3.1 implies that $\mathcal{Y}_z(K) \neq \emptyset$. Thus $\mathcal{Y}_z(K) \subseteq \mathcal{Y}(K) = Y(K) \neq \emptyset$, which completes the proof of Proposition 5.1.4. $\square$

Thus $s(G_K) \subset D_{\tilde{x}}$ for a unique K-rational point $x \in X(K)$ and some extension $\tilde{x}$ of x to $k(X)^{\mathrm{sep}}$. This concludes the proof of Theorem A. It also implies that the abelian portion s^{ab} is the image of both x and z under the composite map $X_c(K_c) \hookrightarrow J_c(K_c) \to \widehat{J_c(K_c)} \hookrightarrow H^1(G_{K_c}, TJ_c)$. This map is injective by Lemma 5.1.1, so we must have $x = z$.

5.2 Proof of Theorem B

In this section we explain how Theorem B is deduced from Theorem A. Let k be a field of characteristic zero that strongly satisfies the conditions of Definition 2.3.1. Let C be a smooth, separated, connected curve over k with function field K. For any finite extension L of K, let C^L denote the normalisation of C in L, and for any smooth relative curve $\mathcal{Y} \to C^L$, let $\mathcal{J}_{\mathcal{Y}} := \mathrm{Pic}^0_{\mathcal{Y}/C^L}$ denote the relative Jacobian of $\mathcal{Y}$. Assume that for any such finite extension L and any such relative curve $\mathcal{Y}$ we have $T\text{Ш}(\mathcal{J}_{\mathcal{Y}}) = 0$.

We will show that for any finite extension L of K and any smooth, projective, geometrically connected (not necessarily hyperbolic) curve X over L, the birational section conjecture holds for X, which is to say that for any section $s : G_L \to G_X$ the image $s(G_L)$ is contained in a decomposition group $D_{\tilde{x}}$ for a unique L-rational point $x \in X(L)$ and some extension $\tilde{x}$ of x to $k(X)^{\mathrm{sep}}$. It suffices to show the existence of such an L-rational point, since its uniqueness follows from [NSW08, Corollary 12.1.3] (see Remark 2.2.3).

Proposition 5.2.1. *With the above notation and hypotheses, let $s : G_L \to G_X$ be a section of G_X. Then s is geometric.*

Proof. As in the proof of Lemma 2.2.5, we may choose a neighbourhood $H \subset G_X$ of the section s such that, denoting by $Y \to X$ the corresponding finite morphism, Y is hyperbolic. We have an isomorphism $H \simeq G_Y$, and s naturally defines a section $s_Y : G_L \to G_Y$. Let $L'|L$ be a finite extension such that $Y(L') \neq \emptyset$, and let $M|L$ be a Galois extension of L containing L'. Then $Y_M(M) \neq \emptyset$, and s_Y restricts to a section $s_{Y_M} : G_M \to G_{Y_M}$ of the absolute Galois group of Y_M.

$$\begin{tikzcd}
1 \arrow[r] & G_{Y_{\bar{L}}} \arrow[r] \arrow[d, equal] & G_{Y_M} \arrow[r] \arrow[d, hook] & G_M \arrow[r] \arrow[d, hook] \arrow[l, bend right, "s_{Y_M}"'] & 1 \\
1 \arrow[r] & G_{Y_{\bar{L}}} \arrow[r] & G_Y \arrow[r] & G_L \arrow[r] \arrow[l, bend right, "s_Y"'] & 1
\end{tikzcd}$$

Let C^M denote the normalisation of C in M, and let $\mathcal{Y} \to C^M$ be a model of Y_M over C^M (see Definition 1.1.4). As in the proof of Proposition 5.1.4, after possibly removing finitely many closed points from C^M we may assume that the closed fibres $\mathcal{Y}_c := \mathcal{Y} \times_{C^M} \operatorname{Spec} k(c)$ of $\mathcal{Y}$ are smooth for all $c \in (C^M)^{\mathrm{cl}}$. Then $\mathcal{Y} \to C^M$ is a smooth relative curve whose generic fibre Y_M is hyperbolic and has at least one M-rational point. Theorem A then implies that $s_{Y_M}(G_M)$ is contained in a decomposition subgroup $D^M_{\tilde{y}} \subset G_{Y_M}$ for a unique M-rational point y of Y_M and some extension $\tilde{y}$ of y to $k(X)^{\mathrm{sep}}$. Note we use a superscript M to emphasise that $D^M_{\tilde{y}}$ is a subgroup of G_{Y_M}.

Since $M|L$ is a Galois extension, G_M is a normal subgroup of G_L, and hence $s_Y(G_L)$ normalises $s_{Y_M}(G_M)$ in G_Y. Indeed, for any $\sigma \in G_L$ we have

$$\begin{aligned}
s_{Y_M}(G_M) = s_Y(G_M) &= s_Y(\sigma^{-1} G_M \sigma) \\
&= s_Y(\sigma)^{-1} s_Y(G_M) s_Y(\sigma) \\
&= s_Y(\sigma)^{-1} s_{Y_M}(G_M) s_Y(\sigma)
\end{aligned}$$

This implies that $s_Y(G_L)$ is contained in the decomposition subgroup $D_{\tilde{y}} \subset G_Y$ of the same valuation $\tilde{y}$ of $k(X)^{\mathrm{sep}}$, whose restriction to $k(Y)$ corresponds to the image y' of y under the projection $Y_M \to Y$. Indeed, for any $\sigma \in G_L$ we have

$$\begin{aligned}
s_{Y_M}(G_M) &= s_Y(\sigma)^{-1} s_{Y_M}(G_M) s_Y(\sigma) \\
&\subset s_Y(\sigma)^{-1} D^M_{\tilde{y}} s_Y(\sigma) \\
&= D^M_{s_Y(\sigma)\cdot\tilde{y}}
\end{aligned}$$

Thus $s_{Y_M}(G_M)$ is contained in $D^M_{\tilde{y}}$ and in $D^M_{s_Y(\sigma)\cdot\tilde{y}}$, which implies that $\tilde{y} = s_Y(\sigma)\cdot\tilde{y}$ [NSW08, Corollary 12.1.3], hence $s_Y(G_L)$ normalises $D^M_{\tilde{y}}$ in G_Y. This means $s_Y(G_L)$ is contained in the normaliser of $D^M_{\tilde{y}}$ in G_Y, which is precisely $D_{\tilde{y}}$.

This implies that $s(G_L)$ is contained in the decomposition subgroup $D_{\tilde{y}} \subset G_X$ of the same valuation $\tilde{y}$ of $k(X)^{\mathrm{sep}}$, whose restriction to $k(X)$ corresponds to the image x of y' in X. The point x is then necessarily L-rational, since $D_{\tilde{y}}$ must map surjectively to G_L. $\square$

This concludes the proof of Theorem B.

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